Cauchy process on half-line

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Linear water waves

Assumptions

- fluid is ideal (inviscid, incompressible, heavy)
- no surface tension
- oscillations are small
- motion is irrotational

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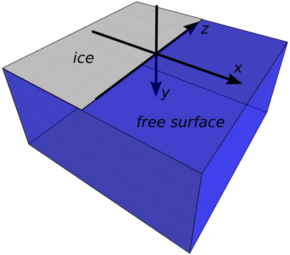
A non-linear wave
(www.surftravelcompany.com/big-wave-pics/mavericks.jpg)

A more linear wave
(art4linux.org/system/files/water-waves-1600.jpg)

Three-dimensional sloshing (1/2)



Velocity at time t is irrotational:

$$\vec{v}_t(x, y, z) = \nabla u_t(x, y, z).$$

u_t — velocity potential

The water is incompressible, so $\mathrm{div}\,\vec{v}_t = 0$, or

$$\Delta u_t = 0 \qquad \text{in the water.}$$

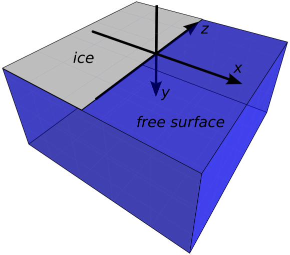
Under the ice $\vec{v}_t$ is tangent to the surface, so

$$\frac{\partial}{\partial n} u_t = 0.$$

On the free surface

$$g\, \frac{\partial}{\partial n} u_t = \frac{\partial^2}{\partial t^2} u_t.$$

Three-dimensional sloshing (2/2)



Harmonic oscilations:

$$u_t(x, y, z) = u(x, y, z)\cos(\omega t).$$

Sloshing problem:

$$\Delta u = 0 \qquad \text{inside,}$$

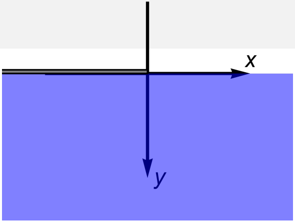
$$\frac{\partial}{\partial n} u = 0 \qquad \text{at the fixed boundary,}$$

$$\frac{\partial}{\partial n} u = \lambda u \qquad \text{on the free surface,}$$

with spectral parameter $\lambda = \frac{\omega^2}{g}$.

Various types of containers have been studied.

Two-dimensional sloshing



Two-dimensional **sloshing problem**:

$$\Delta u(x, y) = 0 \qquad \text{inside,}$$
$$\frac{\partial}{\partial n} u(x, y) = 0 \qquad \text{at the fixed boundary,}$$
$$\frac{\partial}{\partial n} u(x, y) = \lambda u(x, y) \qquad \text{on the free surface.}$$

This corresponds to 3-D oscillations not depending on z.

In half-plane with semi-infinite *dock*:

$$\Delta u(x, y) = 0 \qquad \text{for } x \in \mathbf{R},\ y > 0,$$
$$\frac{\partial}{\partial y} u(x, 0) = 0 \qquad \text{for } x \leq 0,$$
$$\frac{\partial}{\partial y} u(x, 0) = -\lambda u(x, y) \qquad \text{for } x > 0.$$

History in a nutshell

Sloshing problem was studied by:

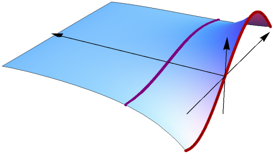
Euler (1761),
Green (1838 and 1842),
Airy (1845),
Ostrogradsky (1862),
Kirchhoff (1879),
Macdonald (1894 and 1896),
Poincaré (1910),
Poisson (1816 and 1828),
Kelland (1840–1844),
Stokes (1846),
Rayleigh (1876 and 1899),
Greenhill (1887),
Chrystal (1905 and 1906),
Hadamard (1910 and 1916).

Sloshing in half-plane with semi-infinite dock:

Friedrichs, Lewy (1947),
Holford (1964).

See also: Chakrabarti, Mandal, Rupanwita Gayen, *The dock problem revisited* (2006)

Cauchy semigroup



If $u(x,y)$ is harmonic in $\{y \geq 0\}$, then

$$u(x,y) = \frac{1}{\pi}\int \frac{y}{y^2+(x-z)^2}\, u(z,0)dz.$$

Hence $u(\cdot,y) = P_y u(\cdot 0)$, where

$$P_y f(x) = \frac{1}{\pi}\int \frac{y}{y^2+(x-z)^2}\, f(z)dz$$

is the **Cauchy semigroup** (or Poisson semigroup).

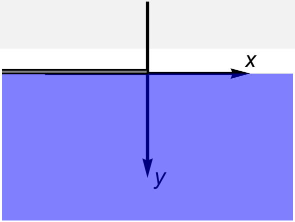
Furthermore,

$$\frac{\partial}{\partial n} u(\cdot,0) = -\frac{\partial}{\partial y} u(\cdot,0) = -\lim_{y\searrow 0} \frac{P_y u(\cdot,0) - u(\cdot,0)}{y} = -\mathcal{A}u(\cdot,0),$$

where

$$\mathcal{A}f(x) = -\sqrt{-\frac{d^2}{dx^2}}\, f(x) = \frac{1}{\pi}\int \frac{f(z)-f(x)}{(z-x)^2}\, dz.$$

Two-dimensional sloshing revisited



Two-dimensional **sloshing problem**
in half-plane with semi-infinite dock:

$$\Delta u(x,y) = 0 \qquad \text{for } x \in \mathbf{R},\ y > 0,$$
$$\frac{\partial}{\partial y}\, u(x,0) = 0 \qquad \text{for } x \le 0,$$
$$\frac{\partial}{\partial y} u(x,0) = -\lambda u(x,y) \qquad \text{for } x > 0.$$

Equivalently: $u(x,y) = P_y \psi(x)$, where

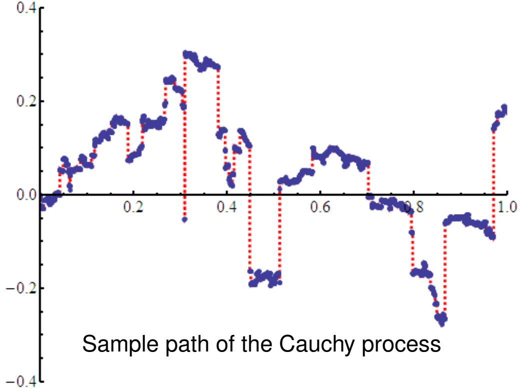
$$\mathcal{A}\psi(x) = 0 \qquad \text{for } x \le 0,$$
$$\mathcal{A}\psi(x) = -\lambda \psi(x) \qquad \text{for } x > 0.$$

Cauchy process (1/2)

P_t are transition operators of the **Cauchy process** X_t.
$\mathcal{A}$ is the infinitesimal generator of X_t.

X_t is the Lévy process with Lévy measure $\frac{1}{\pi}\frac{1}{x^2}\,dx$.
(no Gaussian part, no drift)

X_t is the symmetric
1-stable process.
(unique up to scaling)



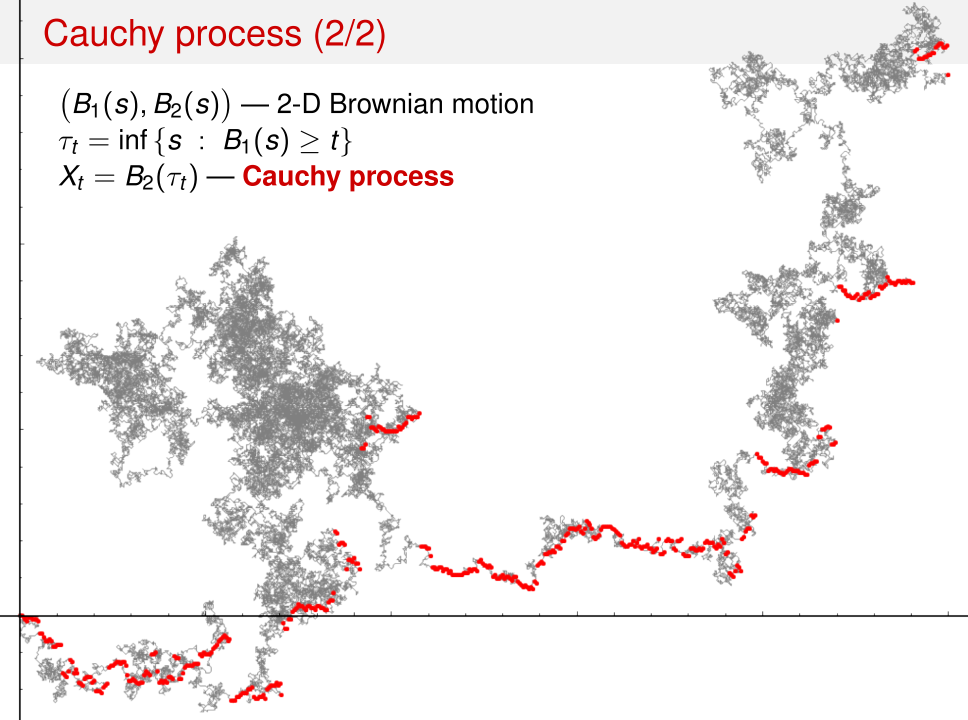
Sample path of the Cauchy process

Cauchy process (2/2)

$\big(B_1(s), B_2(s)\big)$ — 2-D Brownian motion

$\tau_t = \inf\{s \ : \ B_1(s) \geq t\}$

$X_t = B_2(\tau_t)$ — **Cauchy process**



Killed semigroup

Let $D = (0, \infty)$.

Define the first exit time:

$$\tau_D = \inf\big\{t \geq 0 \ : \ X_t \notin D\big\}.$$

Transition operators of the killed process:

$$P_t^D f(x) = \mathbf{E}^x\big[f(X_t) \ ; \ t < \tau_D\big].$$

Question

What are the eigenvalues/eigenfunctions of P_t^D?

What is the spectral representation of P_t^D?

The two problems

Sloshing problem — for a kind of reflected process:

$$\begin{array}{lll|ll} \Delta u(x,y) = 0, & & & \mathcal{A}\psi(x) = 0 & \text{for } x \le 0, \\ \frac{\partial}{\partial y} u(x,0) = 0 & \text{for } x \le 0. & & \mathcal{A}\psi(x) = -\lambda\psi(x) & \text{for } x > 0. \\ \frac{\partial}{\partial y} u(x,0) = -\lambda u(x,0) & \text{for } x > 0 & & & \end{array}$$

Explicit solution — Friedrichs, Levy, 1947.

Eigenfunctions of P_t^D — for the killed process:

$$\begin{array}{ll|ll} \psi(x) = 0 & \text{for } x \le 0, & \Delta u(x,y) = 0 & \\ \mathcal{A}\psi(x) = -\lambda\psi(x) & \text{for } x > 0. & u(x,0) = 0 & \text{for } x \le 0, \\ & & \frac{\partial}{\partial y} u(x,0) = -\lambda u(x,0) & \text{for } x > 0. \end{array}$$

Explicit solution — TK+MK+JM+AS, 2009.

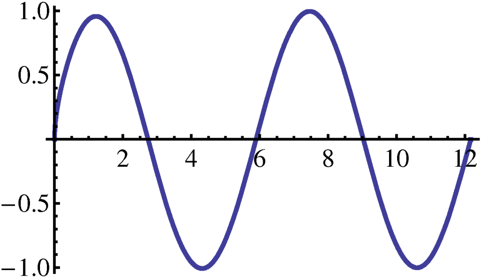
Main result (1/3)

Theorem (TK+MK+JM+AS)

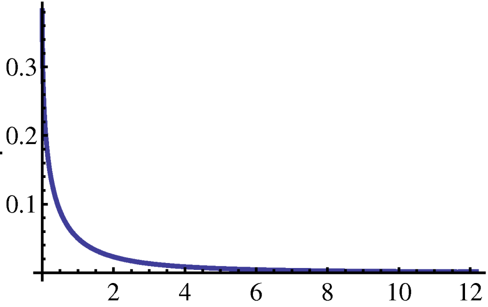
The eigenfunctions of P_t^D are given by

$$\psi_\lambda(x) = \sin(\lambda x + \tfrac{\pi}{8}) - r(\lambda x),$$

$$r(x) = \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{t}{1+t^2} \exp\left(-\frac{1}{\pi} \int_0^\infty \frac{\log(t+s)}{1+s^2} ds \right) e^{-tx} dt.$$



Graph of $\psi_1(x)$



Graph of $r(x) = \sin(x + \frac{\pi}{8}) - \psi_1(x)$

Main result (1/3)

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Fact

The eigenfunctions of the semigroup $\tilde{P}_t^D$ of killed Brownian motion in $(0, \infty)$ *are given by*

$$\tilde{\psi}_\lambda(x) = \sin(\lambda x).$$

Main result (2/3)

| $D = (0,\infty)$

Theorem (TK+MK+JM+AS)

Spectral representation of P_t^D is given by $\Pi : L^2(D) \to L^2((0,\infty))$

$$\Pi f(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\psi_\lambda(x)dx.$$

We have

$$\Pi(P_t^D f)(\lambda) = e^{-\lambda t}\Pi f(\lambda),$$

$$\|\Pi f\|_2 = \|f\|_2, \qquad \text{(Plancherel's theorem)}$$

$$\Pi(\Pi f) = f. \qquad \text{(inversion formula)}$$

Fact

Spectral representation of $\tilde{P}_t^D$ is given by the Fourier sine transform

$$\tilde{\Pi} f(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\tilde{\psi}_\lambda(x)dx.$$

| $\tilde{\psi}_\lambda(x) = \sin(\lambda x)$

$$\tilde{\Pi}(\tilde{P}_t^D f)(\lambda) = e^{-\lambda^2 t}\tilde{\Pi} f(\lambda),$$

$$\|\tilde{\Pi} f\|_2 = \|f\|_2, \qquad \text{(Plancherel's theorem)}$$

$$\tilde{\Pi}(\tilde{\Pi} f) = f. \qquad \text{(inversion formula)}$$

Main result (3/3)

Let $p_t^D(x,y)$ be the kernel of P_t^D. | $D=(0,\infty)$

Theorem (TK+MK+JM+AS)

$$p_t^D(x,y) = \frac{1}{\pi}\frac{t}{t^2+(x-y)^2} - \frac{1}{xy}\int_0^t \frac{f(\frac{s}{x})f(\frac{t-s}{y})}{\frac{s}{x}+\frac{t-s}{y}}\,ds$$

$$\textit{with} \quad f(s) = \frac{1}{\pi}\frac{s}{1+s^2}\exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(s+w)}{1+w^2}\,dw\right).$$

$$\mathbf{P}^x(\tau_D \in dt) = \frac{1}{\pi}\frac{x}{t^2+x^2}\exp\left(\frac{1}{\pi}\int_0^\infty \frac{\log(\frac{t}{x}+w)}{1+w^2}\,dw\right)dt.$$

Proof.

$$p_t^D(x,y) = \frac{2}{\pi}\int_0^\infty e^{-\lambda t}\psi_\lambda(x)\psi_\lambda(y)d\lambda,$$

$$\mathbf{P}^x(\tau_D > t) = \int_D p_t^D(x,y)dy.$$

□

Proof of spectral representation theorem

$$\begin{aligned}\Pi f(\lambda) &= \sqrt{\tfrac{2}{\pi}}\int_0^\infty f(x)\psi_\lambda(x)dx \\ &= \sqrt{\tfrac{2}{\pi}}\int_0^\infty f(x)\sin(\lambda x + \tfrac{\pi}{8})dx - \sqrt{\tfrac{2}{\pi}}\int_0^\infty f(x)r(\lambda x)dx.\end{aligned}$$

- $\Pi : L^2(D) \to L^2(D)$ is bounded (Hardy-Hilbert inequality);
- $\operatorname{supp} f \cap \operatorname{supp} g = \varnothing$ implies $\langle \Pi f, \Pi g\rangle = 0$ (eigenfunctions are orthogonal);
- $A \mapsto \langle \Pi \mathbf{1}_A, \Pi \mathbf{1}_A \rangle$ is a measure;
- it scales well, so it is Lebesgue, up to a constant;
- Π is unitary, up to a constant;
- the constant is one (by considering $\mathbf{1}_{(1,1+\varepsilon)}$).

Derivation of ψ_λ (1/2)

Problem:

$$(1)\ \Delta u(x,y) = 0$$

$$(2)\ \frac{\partial}{\partial y} u(x,0) = 0 \qquad \text{for } x \le 0,$$

$$(3)\ \frac{\partial}{\partial n} u(x,0) = -\lambda u(x,0) \qquad \text{for } x > 0;$$

- $e^{k\lambda x}\sin(k\lambda y)$ satsifies (1) and (2),
  $e^{k\lambda x}\sin(k\lambda y - \arctan k)$ satisfies (1) and (3),
  $e^{-\lambda y}\sin(\lambda x + \vartheta)$ satisfies (1) and (3);
- try the following form of the solution

$$u(x,y) = \int_0^\infty w(k) e^{k\lambda x}\sin(k\lambda y)\,dk \qquad \text{for } x \le 0,\ y \ge 0,$$

$$u(x,y) = e^{-\lambda y}\sin(\lambda x + \vartheta) - \int_{-\infty}^0 w(k) e^{k\lambda x}\sin(k\lambda y - \arctan k)\,dk; \qquad \text{for } x \le 0,\ y \ge 0$$

Derivation of ψ_λ (2/2)

- switch to complex functions: $u = \mathrm{Im}\, F$,
$$F(x+iy) = \int_0^\infty w(k) e^{k\lambda(x+iy)}\, dk \qquad \text{for } x \le 0,\ y \ge 0,$$
$$F(x+iy) = e^{\lambda i(x+iy)+i\vartheta} \qquad \text{for } x \le 0,\ y \ge 0$$
$$- \int_{-\infty}^0 w(k) e^{k\lambda(x+iy) - i \arctan k}\, dk;$$
- both formulas must agree on $x = 0$, $y \ge 0$;
- for $y \ge 0$,
$$\int_0^\infty w(k) e^{ik\lambda y}\, dk = e^{-\lambda y + i\vartheta}$$
$$- \int_{-\infty}^0 w(k) e^{ik\lambda y - i\arctan k}\, dk;$$
- $w(k) e^{i \arctan k_-}$ has Fourier transform $e^{s+i\vartheta}$ for $s \le 0$;
- $w(k)$ is real for all k;
- solve Riemann-Hilbert problem to get the formula for w;
- only for $\vartheta = \frac{\pi}{8}$ the solution u is bounded.


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