

# Fractional Laplace operator and operators with unimodal and isotropic kernels

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Ten definitions  
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Marcel Riesz  
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New formulae  
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Boundary behaviour  
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Isotropic unimodal  
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# Fractional Laplace operator

## Question

How do we define  $(-\Delta)^{\alpha/2}$  for  $\alpha \in (0, 2)$ ?

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# Pointwise definitions

(1) As a singular integral:

$$-(-\Delta)^{\alpha/2} u(x) = c \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d \setminus B(0, \varepsilon)} \frac{u(x+y) - u(x)}{|y|^{d+\alpha}} dy;$$

(2) As a Dynkin's characteristic operator:

$$-(-\Delta)^{\alpha/2} u(x) = c \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d \setminus B(0, \varepsilon)} \frac{u(x+y) - u(x)}{|y|^d (|y|^2 - \varepsilon^2)^{\alpha/2}} dy.$$

(3) Through harmonic extensions, which can be reduced to:

$$-(-\Delta)^{\alpha/2} u(x) = c \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d} \frac{u(x+y) - u(x)}{(\varepsilon^2 + |y|^2)^{(d+\alpha)/2}} dy.$$

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# Operator approach

- (4) As a generator of a semigroup of Markov operators:

$$-(-\Delta)^{\alpha/2} u(x) = c \lim_{t \rightarrow 0^+} \frac{e^{-t(-\Delta)^{\alpha/2}} u(x) - u(x)}{t};$$

here  $e^{-t(-\Delta)^{\alpha/2}} u = p_t * u$ , where  $\hat{p}_t(\xi) = e^{-t|\xi|^\alpha}$ .

- (5) By Bochner's subordination:

$$-(-\Delta)^{\alpha/2} u = \frac{1}{|\Gamma(-\frac{\alpha}{2})|} \int_0^\infty (u - e^{t\Delta} u) t^{-1-\alpha/2} dt;$$

here  $e^{t\Delta}$  is the usual heat operator.

- (6) By Balakrishnan's formula:

$$-(-\Delta)^{\alpha/2} u = \frac{\sin \frac{\alpha\pi}{2}}{\pi} \int_0^\infty \Delta(s - \Delta)^{-1} u s^{\alpha/2-1} ds.$$

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# Riesz potentials, Fourier transform

(7) As the inverse of the Riesz potential operator:

$$u(x) = c \int_{\mathbb{R}^d} \frac{(-\Delta)^{\alpha/2} u(x+y)}{|y|^{d-\alpha}} dy.$$

(8) By Fourier transform:

$$((- \Delta)^{\alpha/2} u) \hat{ }(\xi) = |\xi|^{\alpha} \hat{u}(\xi).$$

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# Duality

(9) Distributional definition:

$$\int_{\mathbb{R}^d} (-\Delta)^{\alpha/2} u(x)v(x)dx = \int_{\mathbb{R}^d} u(x)(-\Delta)^{\alpha/2} v(x)dx$$

for  $v \in C_c^\infty$ .

(10) Via quadratic form:

$$\langle (-\Delta)^{\alpha/2} u, v \rangle = \mathcal{E}(u, v),$$

where

$$\begin{aligned}\mathcal{E}(u, v) &= c \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{d+\alpha}} dx dy \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\xi|^\alpha \hat{u}(\xi) \overline{\hat{v}(\xi)} d\xi.\end{aligned}$$

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# Equivalence

Theorem (numerous authors; see [K–17])

The above ten definitions are all equivalent on  $C_0$ ,  $C_{\text{bu}}$  and  $L^p$ ,  $p \in [1, \infty)$  (whenever meaningful).

- All limits are understood in the corresponding norm, etc.
- Natural limitations:
  - ▶ Riesz potentials work only in  $L^p$ ,  $p \in [1, \frac{d}{\alpha})$ ;
  - ▶ Fourier transform is only defined on  $L^p$ ,  $p \in [1, 2]$ ;
  - ▶ quadratic form is restricted to  $L^2$ .
- Major part was known, but scattered in literature;  
some pieces were apparently missing.
- Very well-known for smooth functions.

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## Remarks

- In  $C_0$  and  $C_{\text{bu}}$ , pointwise convergence to a continuous function implies norm convergence.
- In  $L^p$ , norm convergence implies convergence almost everywhere.
- There are even more definitions! For example:

$$-(-\Delta)^{\alpha/2} u(x) = c \int_{\mathbb{R}^d} \frac{u(x+y) - u(x) - y \cdot \nabla u(x) \mathbb{1}_B(y)}{|y|^{d+\alpha}} dy,$$

$$-(-\Delta)^{\alpha/2} u(x) = c \int_{\mathbb{R}^d} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{d+\alpha}} dy,$$

or the definition involving viscosity solutions.

- Pointwise definitions are to some extent equivalent (e.g. for smooth functions), but not in full generality.

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# Caffarelli–Silvestre extension technique

- ‘Harmonic extension’ refers to the representation of  $-(-\Delta)^{\alpha/2}$  as a Dirichlet-to-Neumann operator:

$$\begin{cases} \Delta_x u(x, y) + c y^{2-2/\alpha} \frac{\partial^2 u}{\partial y^2}(x, y) = 0 & \text{for } y > 0 \\ u(x, 0) = f(x) \\ \partial_y u(x, 0) = -(-\Delta)^{\alpha/2} f(x) \end{cases}$$

- (Re-)discovered and popularised in [Caffarelli–Silvestre–07], but occasionally used already in 1960s.
- Remark: if  $c y^{2-2/\alpha}$  is replaced by  $A(dy)$ , one obtains  $-\psi(-\Delta)$  for a complete Bernstein function  $\psi$ , and  $A \mapsto \psi$  is a bijection; see [K–Mucha–18<sup>+</sup>].

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# Origins

- In 1938, M. Riesz published his article *Intégrales de Riemann–Liouville et potentiels.*
- It contains a lot of results!  
Some of them are often attributed to other authors.
- M. Riesz considered what is now called **Riesz potential**:

$$I_\alpha u(x) = c \int_{\mathbb{R}^d} \frac{u(x-y)}{|y|^{d-\alpha}} dy,$$

where  $\alpha \in (0, d)$ .

- This extends the classical Newtonian potential, which corresponds to  $\alpha = 2$ .

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# Potential theory

- In modern PDE language, M. Riesz constructed the ‘solution’ of

$$\begin{cases} -(-\Delta)^{\alpha/2} u(x) = -g(x) & \text{for } x \in D, \\ u(x) = f(x) & \text{for } x \in \mathbb{R}^d \setminus D. \end{cases}$$

- The ‘solution’ is given in terms of the **Green function** and the **Poisson kernel**, fundamental objects in potential theory:

$$u(x) = \int_{\mathbb{R}^d \setminus D} P_D(x, z) f(z) dz + \int_D G_D(x, y) g(y) dy.$$

- The Poisson kernel is constructed in a potential-theoretic way, and the Green function is defined by

$$G_D(x, y) = \frac{c}{|y - x|^{d-\alpha}} - \int_{\mathbb{R}^d \setminus D} P_D(x, z) \frac{c}{|y - z|^{d-\alpha}} dz.$$

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# Poisson kernel of a ball

- If  $D = B(0, r)$  is a ball, then

$$P_D(x, z) = c \left( \frac{r^2 - |x|^2}{|z|^2 - r^2} \right)^{\alpha/2} \frac{1}{|x - z|^d}.$$

- The original work of M. Riesz required that  $\alpha < d$ .
- Extension to  $\alpha \geq d$  was carried out in [Kac–57] and [Blumenthal–Getoor–Ray–61].
- A simple corollary: (scale-invariant) Harnack inequality.
- Another simple corollary:

$$\begin{aligned} -(-\Delta)^{\alpha/2} [(r^2 - |x|^2)_+^{\alpha/2}] &= c && \text{in } B(0, r), \\ -(-\Delta)^{\alpha/2} [(x_d)_+^{\alpha/2}] &= 0 && \text{in } \mathbb{R}^{d-1} \times (0, \infty). \end{aligned}$$

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# Green function of a ball

- If  $D = B(0, r)$  is a ball, then

$$G_D(x, y) = \frac{c}{|y - x|^{d-\alpha}} - \int_{\mathbb{R}^d \setminus D} P_D(x, z) \frac{c}{|y - z|^{d-\alpha}} dz.$$

- [Blumenthal–Getoor–Ray–61] simplified this to

$$\begin{aligned} G_D(x, y) &= c \frac{1}{|x - y|^{d-\alpha}} \int_0^{S(r, x, y)} \frac{s^{\alpha/2-1}}{(1+s)^{d/2}} ds \\ &= c \frac{(S(r, x, y))^{\alpha/2}}{|x - y|^{d-\alpha}} {}_2F_1\left(\begin{array}{c} \frac{d}{2}, \frac{\alpha}{2} \\ 1 + \frac{\alpha}{2} \end{array} \middle| -S(r, x, y)\right), \end{aligned}$$

where

$$S(r, x, y) = \frac{(r^2 - |x|^2)(r^2 - |y|^2)}{r^2|x - y|^2}.$$

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# Meijer G-function

Theorem ([Dyda–Kuznetsov–K–17a])

$$\begin{aligned} -(-\Delta)^{\alpha/2} \left[ G_{p,q}^{m,n} \left( \underbrace{\overbrace{a_1, \dots, a_p}^{\mathbf{a}}, \underbrace{b_1, \dots, b_q}_\mathbf{b} \mid |x|^2 \right) \right] &= \\ &= -2^\alpha G_{p+2,q+2}^{m+1,n+1} \left( \begin{matrix} \frac{1-d-\alpha}{2}, & \mathbf{a} - \frac{\alpha}{2}, & -\frac{\alpha}{2} \\ 0, & \mathbf{b} - \frac{\alpha}{2}, & 1 - \frac{d}{2} \end{matrix} \mid |x|^2 \right). \end{aligned}$$

- The Meijer G-function  $G_{p,q}^{m,n}$  is not nice to work with.
- A lot of functions can be written in terms of  $G_{p,q}^{m,n}$ .
- [Prudnikov–Brychkov–Marichev–90] contains a 100-page-long table of such expressions.

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## Special cases: full space (1/2)

- Generalised hypergeometric function is somewhat simpler:

$$\begin{aligned} -(-\Delta)^{\alpha/2} \left[ {}_pF_q \left( \underbrace{\overbrace{a_1, \dots, a_p}^{\mathbf{a}}, \underbrace{b_1, \dots, b_{q-1}}^{\mathbf{b}}}_{\mathbf{b}} \mid \frac{d}{2} \right) \right] &= \\ = -2^\alpha \frac{\Gamma(\mathbf{a} + \frac{\alpha}{2})\Gamma(\mathbf{b})}{\Gamma(\mathbf{a})\Gamma(\mathbf{b} + \frac{\alpha}{2})} {}_pF_q \left( \mathbf{a} + \frac{\alpha}{2} \mid \frac{d}{2} \right) & - |x|^2. \end{aligned}$$

- A more explicit example is:

$$\begin{aligned} -(-\Delta)^{\alpha/2} [|x|^p (1 + |x|^2)^{q/2}] &= \\ = -\frac{2^\alpha}{\Gamma(-\frac{q}{2})} G_{3,3}^{2,2} \left( \begin{matrix} 1 - \frac{d+\alpha}{2}, & 1 + \frac{p+q-\alpha}{2}, & -\frac{\alpha}{2} \\ 0, & \frac{p-\alpha}{2}, & 1 - \frac{d}{2} \end{matrix} \mid |x|^2 \right). & \end{aligned}$$

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## Special cases: full space (2/2)

- We recover formulae from [Samko–01], for example

$$-(-\Delta)^{\alpha/2}[\exp(-|x|^2)] = c {}_1F_1\left(\frac{d+\alpha}{2}; \frac{d}{2}; -|x|^2\right)$$

and

$$\begin{aligned} -(-\Delta)^{\alpha/2} & \left[ \frac{1}{(1+|x|^2)^{(d-\alpha)/2+n}} \right] \\ &= -c \frac{{}_2F_1\left(-n, -\frac{\alpha}{2}; 1 - \frac{d-\alpha}{2} - n; 1 + |x|^2\right)}{(1+|x|^2)^{(d+\alpha)/2+n}}. \end{aligned}$$

- By the way, when  $n = 0$ , we obtain

$$-(-\Delta)^{\alpha/2} \left[ \frac{1}{(1+|x|^2)^{(d-\alpha)/2}} \right] = -\frac{c}{(1+|x|^2)^{(d+\alpha)/2}}.$$

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## Special cases: unit ball

- Our result also includes the expressions found in [Biler–Imbert–Karch–11] and [Dyda–12]:

$$-(-\Delta)^{\alpha/2} \left[ (1 - |x|^2)_+^p \right] = -c {}_2F_1\left(\frac{d+\alpha}{2}, \frac{\alpha}{2} - p; \frac{d}{2}; |x|^2\right)$$

in the unit ball  $B(0, 1)$ .

- More generally: in the unit ball  $B(0, 1)$  we have

$$\begin{aligned} -(-\Delta)^{\alpha/2} \left[ (1 - |x|^2)_+^{\alpha/2} P_n^{(\frac{\alpha}{2}, \frac{d}{2}-1)}(2|x|^2 - 1) \right] &= \\ &= -c P_n^{(\frac{\alpha}{2}, \frac{d}{2}-1)}(2|x|^2 - 1), \end{aligned}$$

where  $P_n^{(\alpha, \beta)}(r)$  is the Jacobi polynomial.

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## Application: eigenvalues in the unit ball

- The eigenvalue problem for  $(-\Delta)^{\alpha/2}$  in the unit ball:

$$\begin{cases} (-\Delta)^{\alpha/2}\varphi(x) = \lambda\varphi(x) & \text{for } x \in B(0, 1), \\ \varphi(x) = 0 & \text{for } x \in \mathbb{R}^d \setminus B(0, 1). \end{cases}$$

- Let  $\lambda_{d,0} \leqslant \lambda_{d,1} \leqslant \lambda_{d,2} \leqslant \dots$  be the sequence of those eigenvalues which correspond to radial eigenfunctions.
- Upper bounds for  $\lambda_{d,n}$ : Rayleigh–Ritz variational formula.
- Lower bounds for  $\lambda_{d,n}$ : Weinstein–Aronszajn method of intermediate problems.
- In both methods we use  $(1 - |x|^2)_+^{\alpha/2} P_n^{(\frac{\alpha}{2}, \frac{d}{2}-1)}(2|x|^2 - 1)$  as test functions.
- This leads to both numerical and analytical bounds;  
see [Dyda–Kuznetsov–K–17b].

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# Harmonic polynomials

- Bochner's relation implies that if  $V(x)$  is a solid harmonic polynomial of degree  $\ell$ , then:

$$-(-\Delta)^{\alpha/2} [V(x) f(|x|)] = V(x) g(|x|) \quad \text{in } \mathbb{R}^d$$

if and only if

$$-(-\Delta)^{\alpha/2} [f(|y|)] = g(|y|) \quad \text{in } \mathbb{R}^{d+2\ell}.$$

- Here 'solid' = 'homogeneous'; examples of  $V(x)$ :  
 $1, x_1, x_1 x_2, x_1 x_2 \dots x_d, x_1^2 - x_2^2$ .
- Corollary:  $\lambda_{d+2\ell,n}$  are the eigenvalues of  $(-\Delta)^{\alpha/2}$  in  $B(0, 1)$ .
- Our bounds allow us to compare  $\lambda_{d,1}$  and  $\lambda_{d+2,0}$  for small  $d$ .

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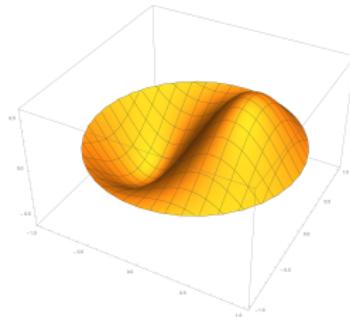
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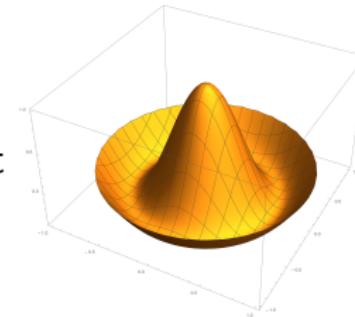
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# Kulczycki's conjecture

- Let  $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2$  be the sequence of all eigenvalues of  $(-\Delta)^{\alpha/2}$  in the unit ball  $B(0, 1)$ .
- It is easy to see that  $\lambda_0$  corresponds to the ground state, the unique positive (and radial) eigenfunction.
- Kulczycki conjectured that  $\lambda_1$  corresponds to an antisymmetric eigenfunction:



and not



- Numerical bounds strongly support the conjecture, and analytical bounds prove it when  $d \leqslant 2$  (or  $\alpha = 1$ ,  $d \leqslant 9$ ).

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# Half-line

Theorem ([K-11], [K-Małecki–Ryznar–13], [Kuznetsov–K–18])

For  $\lambda > 0$ , the solution of

$$\begin{cases} (-\Delta)^{\alpha/2} \varphi(x) = \lambda^\alpha \varphi(x) & \text{for } x > 0, \\ \varphi(x) = 0 & \text{for } x \leq 0 \end{cases}$$

is given by

$$\varphi_\lambda(x) = \sin\left(\lambda x + \frac{(2-\alpha)\pi}{8}\right) - \int_0^\infty e^{-\lambda xs} \Phi(s) ds,$$

where

$$\begin{aligned} \Phi(s) &= \frac{\sqrt{2\alpha} \sin \frac{\alpha\pi}{2}}{2\pi} \frac{s^\alpha}{1 + s^{2\alpha} - 2s^\alpha \cos \frac{\alpha\pi}{2}} \\ &\times \exp\left(\frac{1}{\pi} \int_0^\infty \frac{1}{1+r^2} \log \frac{1-s^2r^2}{1-s^\alpha r^\alpha} dr\right). \end{aligned}$$

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# Half-line and interval

$$\varphi_\lambda(x) = \sin\left(\lambda x + \frac{(2-\alpha)\pi}{8}\right) - \int_0^\infty e^{-\lambda xs} \Phi(s) ds,$$

- The function  $\Phi(s)$  can be expressed in terms of the Koyama–Kurokawa's double sine function  $S_2(\alpha; s)$ .
- Asymmetric operators are included in [Kuznetsov–K–18].
- Eigenfunctions in the interval  $(-1, 1)$  can be approximated by glueing together  $\varphi_\lambda(1+x)$  and  $\varphi_\lambda(1-x)$  near  $x = 0$ .

## Theorem ([K–12])

The eigenvalues  $\lambda_n$  of  $(-\Delta)^{\alpha/2}$  in  $(-1, 1)$  satisfy

$$\lambda_n = \left( \frac{n\pi}{2} - \frac{(2-\alpha)\pi}{8} \right)^{\alpha/2} + O\left(\frac{1}{n}\right).$$

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# Harmonicity

- $u$  is  $\alpha$ -harmonic in  $D$  if  $u$  is continuous in  $D$  and

$$-(-\Delta)^{\alpha/2} u(x) = 0 \quad \text{for } x \in D;$$

- equivalently (see [Bogdan–Byczkowski–99]):

$$u(x) = \int_{\mathbb{R}^d \setminus B} P_B(x, z) u(z) d$$

for any ball  $B$  (equivalently: any bounded open  $B$ )  
such that  $\overline{B} \subseteq D$ .

- $u$  is regular  $\alpha$ -harmonic in  $D$  if

$$u(x) = \int_{\mathbb{R}^d \setminus D} P_D(x, z) u(z) dz.$$

- If  $u \in C(\mathbb{R}^d \setminus D)$  and  $D$  has the exterior cone property,  
'regular  $\alpha$ -harmonic' means ' $\alpha$ -harmonic and  $C(\mathbb{R}^d)$ '.

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# Harmonic functions in a ball

Theorem ([Hmissi–94], [Bogdan–99], [Chen–Song–98])

Non-negative  $\alpha$ -harmonic functions in the unit ball  $D = B(0, 1)$  have the form

$$u(x) = \int_{\mathbb{R}^d \setminus D} P_D(x, z)f(z)dz + \int_{\partial D} M_D(x, z)\mu(dz),$$

where the Poisson kernel and the Martin kernel are given by

$$P_D(x, z) = c \left( \frac{r^2 - |x|^2}{|z|^2 - r^2} \right)^{\alpha/2} \frac{1}{|x - z|^d},$$

$$M_D(x, z) = \frac{(r^2 - |x|^2)^{\alpha/2}}{|x - z|^d}.$$

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New formulae  
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Boundary behaviour  
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# Harmonic functions in any open set

Theorem ([Bogdan–99], [Chen–Song–98], [Song–Wu–99],  
[Bogdan–Kulczycki–K–08])

Non-negative  $\alpha$ -harmonic functions in an arbitrary open set  $D$  have the form

$$u(x) = \int_{\mathbb{R}^d \setminus D} P_D(x, z) f(z) dz + \int_{\partial_m D} M_D(x, z) \mu(dz),$$

where the Poisson kernel and the Martin kernel are given by

$$P_D(x, z) = \int_D G_D(x, y) \frac{c}{|y - z|^{d+\alpha}} dy,$$

$$M_D(x, z) = \lim_{\substack{y \rightarrow z \\ y \in D}} \frac{G_D(x, y)}{G_D(x_0, y)}$$

and  $\partial_m D = \{z \in \partial D : P_D(x_0, z) = \infty\}$ .

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# Boundary Harnack inequality

- The existence of  $M_D(x, z)$  is a consequence of the following boundary Harnack inequality.

Theorem ([Bogdan–97], [Song–Wu–99],  
[Bogdan–Kulczycki–K–08])

Let  $D$  be an open set,  $z \in \partial D$ . If  $u$  is regular  $\alpha$ -harmonic in  $D$  and  $u(x) = 0$  for  $x \in B(z, r) \setminus D$ , then

$$\lim_{\substack{x \rightarrow z \\ x \in D}} \frac{u(x)}{\int_D G_D(x, y) dy} \text{ exists}$$

(uniformly with respect to  $u$ ,  $D$  and  $z$ ).

- No regularity of  $D$  is needed.
- The denominator is not  $\alpha$ -harmonic.

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# Harmonic functions and potentials

- A simple corollary: if  $g$  is nonnegative and

$$u(x) = \int_D G_D(x, y)g(y)dy,$$

then  $u(x) \asymp \int_D G_D(x, y)dy$  (unless  $u(x) \equiv 0$ ).

- In fact, if  $g$  is nonnegative and continuous at  $z \in \partial D$ , then

$$\lim_{\substack{x \rightarrow z \\ x \in D}} \frac{u(x)}{\int_D G_D(x, y)dy} \quad \text{exists.}$$

- If  $D = B(0, r)$  is a ball, then

$$\int_D G_D(x, y)dy = c(r^2 - |x|^2)_+^{\alpha/2}.$$

- If  $D$  has  $C^{1,1}$  boundary, then

$$\int_D G_D(x, y)dy \sim c(\text{dist}(x, \partial D))^{\alpha/2}.$$

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## Translation-invariant operators

- Consider a symmetric function  $\nu$  on  $\mathbb{R}^d$  such that  $\int_{\mathbb{R}^d} \min\{1, |x|^2\} \nu(x) dx < \infty$ , and let

$$Lu(x) = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^d \setminus B(0, \varepsilon)} (u(x+y) - u(x)) \nu(y) dy.$$

- By general theory, a 'solution' of

$$\begin{cases} Lu(x) = -g(x) & \text{for } x \in D, \\ u(x) = f(x) & \text{for } x \in \mathbb{R}^d \setminus D. \end{cases}$$

is given in terms of the Green function and the Poisson kernel:

$$u(x) = \int_{\mathbb{R}^d \setminus D} f(z) P_D(x, dz) + \int_D g(y) G_D(x, dy).$$

- In fact  $L$  need not be translation-invariant: the above is true whenever  $L$  is the generator of a **Markov process**.
- No hope for detailed results in this generality.

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# Isotropic unimodal kernels

- An isotropic unimodal kernel is a function of the form  $\nu(|x|)$  for a non-increasing  $\nu$ .
- Why this class of kernels?
  - ▶ The convolution of two isotropic unimodal kernels is again isotropic unimodal.
  - ▶ Tauberian theorems require some kind of monotonicity.
- A large part of the theory for the fractional Laplace operator  $-(-\Delta)^{\alpha/2}$  extends to operators  $L$  with isotropic and unimodal kernels  $\nu(x) = \nu(|x|)$ .
- Study initiated in [Grzywny–14] (extending previous works on narrower classes of operators).

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# Regular variation

- A function  $f$  is regularly varying at  $\infty$  with index  $\alpha$  if

$$\lim_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha \quad \text{for } \lambda > 1.$$

- More generally,  $f$  is  $O$ -regularly varying (*OR*) at  $\infty$  with lower index  $\alpha$  and upper index  $\beta$  if

$$\lambda^\alpha \leq \liminf_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)}, \quad \limsup_{x \rightarrow \infty} \frac{f(\lambda x)}{f(x)} \leq \lambda^\beta$$

for  $\lambda > 1$ .

- $O$ -regular variation at 0 is defined in a similar way.

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## Pruitt's functions

- Karamata's Tauberian theory tells us, for example, that

$f$  is OR at 0 with lower index  $\alpha > -1$

$$\int_0^x f(y)dy \asymp x f(x).$$

- Following [Pruitt-81], we define

$$K(r) = \int_{B(0,r)} \frac{|x|^2}{r^2} \nu(|x|)dx,$$

$$L(r) = \int_{\mathbb{R}^d \setminus B(0,r)} \nu(|x|)dx.$$

- Thus,

$$K(r) \asymp r^d \nu(r) \iff \nu(r) \text{ is OR at 0 with } \alpha > -d - 2.$$

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# Estimates

## Theorem ([Grzywny–K–18])

If  $d \geq 3$ , then the potential kernel  $U(|x|)$  (the kernel of  $-L^{-1}$ ) satisfies

$$c \frac{\nu(r)}{(K(r) + L(r))^2} \leq U(r) \leq c \frac{r^{-d} K(r)}{(K(r) + L(r))^2}$$

The Green function of a ball  $B(0, r)$  satisfies

$$G_{B(0,r)}(x, y) \leq c \frac{|x - y|^{-d} K(|x - y|)}{(K(r) + L(r))^2},$$

$$G_{B(0,r)}(x, y) \geq c \frac{\nu(|x - y|)}{(K(r_x) + L(r_x))(K(r_y) + L(r_y))},$$

where  $r_x = \min(|x - y|, r - |x|)$  and  $r_y = \min(|x - y|, r - |y|)$ .

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# A priori estimate

## Theorem ([Grzywny–K–18])

A non-negative  $L$ -harmonic function  $u$  in  $B(0, r)$  satisfies

$$u(x) \leq c \frac{r^{-d} K(r)}{K(r) + L(r)} \int_{\mathbb{R}^d \setminus B(0, r/2)} f(z) dz.$$

- The proof is quite explicit: we average  $P_{B(0, \varrho)}(0, z)$  over  $\varrho \in (0, r)$  to get a bounded kernel which reproduces harmonic functions.

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# Regularity

## Theorem ([Grzywny–K–18])

If  $\nu(|x|)$  is  $C^k(\mathbb{R}^d \setminus B(0, \varepsilon))$  for all  $\varepsilon > 0$ , then  $L$ -harmonic functions in  $D$  are  $C^k(D)$ .

- Higher regularity of  $u$  is expected for kernels sufficiently singular near 0.
- Any solution of

$$\begin{cases} Lu(x) = -g(x) & \text{for } x \in D, \\ u(x) = f(x) & \text{for } x \in \mathbb{R}^d \setminus D \end{cases}$$

is a sum of an  $L$ -harmonic function and  $U * g$ .

- This allows one to study the regularity of  $u$  in terms of the regularity of  $g$  and the characteristics of  $\nu$ .

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## Boundary estimate

Theorem ([Bogdan–Kumagai–K–15], [Juszczyszyn–K–18],  
[Kim–Song–Vondraček–18<sup>abc</sup>], [Grzywny–K–18])

Suppose that  $\nu(r)$  is OR at 0 with  $\alpha > -d - 2$  and

$$\lim_{\delta \rightarrow 0^+} \frac{\nu(r + \delta)}{\nu(r)} = 1 \quad \text{uniformly in } r > 1.$$

Then the boundary Harnack inequality holds:

Let  $D$  be an open set,  $z \in \partial D$ . If  $u$  is regular  $L$ -harmonic in  $D$  and  $u(x) = 0$  for  $x \in B(z, r) \setminus D$ , then

$$\lim_{\substack{x \rightarrow z \\ x \in D}} \frac{u(x)}{\int_D G_D(x, y) dy} \quad \text{exists}$$

(uniformly with respect to  $u$ ,  $D$  and  $z$ ).

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