Spectral Theory for Subordinate Brownian Motions in Half-line

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Outline

K., 2010
Spectral analysis of subordinate Brownian motions in half-line

K., Jacek Małecki, Michał Ryznar, 2011
Suprema of Lévy processes

Part I

Introduction

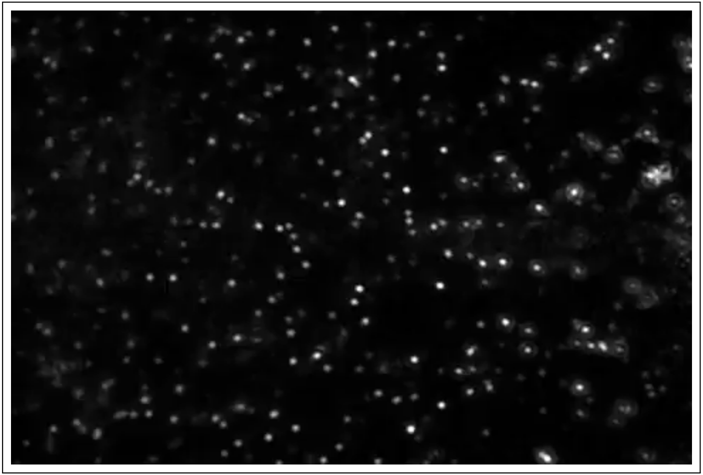
- Brownian motion
- Lévy processes
- Warm-up: Brownian motion in an interval
- Motivation: Brownian motion in half-line
- Complete monotonicity, complete Bernstein functions and subordinate Brownian motions

Part I

Section 1

Brownian motion

Brownian motion



Source: YouTube, `http://youtube.com/watch/?v=cDcprgWiQEY`

- X_t is the position of the particle at time $t \geq 0$

Brownian motion: mathematical model

Definition

The **Brownian motion** (**BM**) is a stochastic process X_t with the following properties:

- $X_0 = x$
- independent increments:

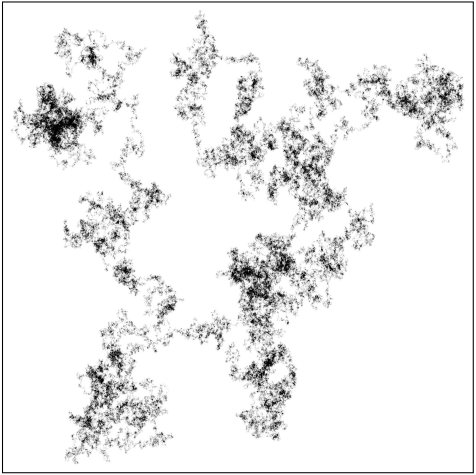
$$0 \le t_0 < t_1 < \ldots < t_n,$$

$$\Downarrow$$

$$X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1},\ \ldots,\ X_{t_n} - X_{t_{n-1}} \text{ are independent}$$

- stationarity: law of $X_t - X_s$ depends only on $t - s$
- isotropy: law of $X_t - X_0$ is invariant under rotations
- continuity of paths: $t \mapsto X_t$ is continuous

Brownian motion: simulation



Source: Wikipedia, `http://en.wikipedia.org/wiki/File:2D_Random_Walk_400x400.ogv`

Brownian motion and PDEs (1)

Central limit theorem

Brownian motion is a Gaussian process:
$(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ has Gaussian distribution. $\square$

Notation

- $\mathbf{P}_x$, $\mathbf{E}_x$ correspond to process starting at x
- $\mathbf{E}_x(Z; E) = \int_E Z d\mathbf{P}_x$
- For readability, we use both X_t and $X(t)$

- Components of X_t are independent
- Variance of each component is $2ct$ for some $c > 0$
- Typically, $c = \frac{1}{2}$, but we take $c = 1$

Brownian motion and PDEs (2)

Theorem

The function:

$$u(t,x) = \mathbf{E}_x f(X_t)$$

solves the **heat equation**:

$$\frac{\partial u}{\partial t}(t,x) = c\,\Delta u(t,x), \quad u(0,x) = f(x) \qquad \square$$

- $\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_d}\right)^2$

Brownian motion and PDEs (3)

Definition

For D open, we define the **first exit time**:

$$\tau_D = \inf\{t \geq 0 : X_t \notin D\}$$

Theorem (Doob, Dynkin, Hunt, Feller, Kakutani, ...)

The function:

$$u(t,x) = \mathbf{E}_x(f(X_t); t < \tau_D)$$

solves **heat equation** in D with **boundary condition**:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= c\,\Delta u(t,x) && (t \geq 0,\ x \in D) \\ u(t,x) &= 0 && (t \geq 0,\ x \in \partial D) \\ u(0,x) &= f(x) && (x \in D) \end{aligned}$$

□

Part I

Section 2

Lévy processes

Lévy processes

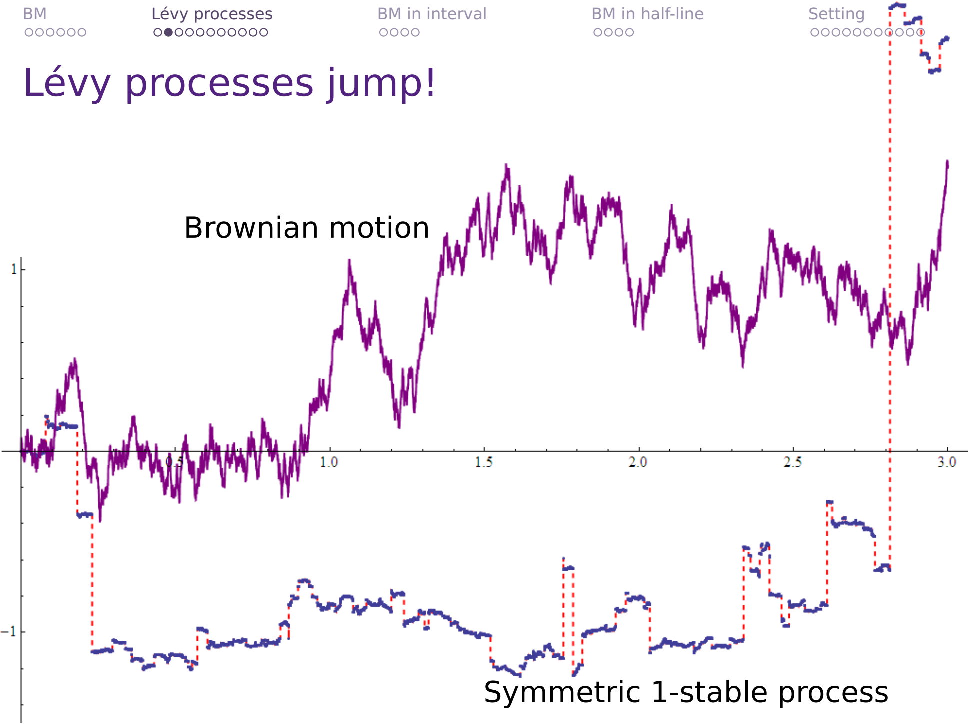
Definition

A **Lévy process** is a stochastic process X_t with the following properties:

- $X_0 = x$
- independent increments
- stationarity
- **no isotropy** (though we will need it later)
- **càdlàg paths**: right-continuous with left limits (instead of continuous)

- We will only study **one-dimensional** Lévy processes
- In dimension one, isotropy $=$ symmetry

Lévy processes jump!



Lévy measure

Definition

The **Lévy measure** ν describes intensity of jumps:

$$\nu(B) = \lim_{t \to 0^+} \frac{\mathbf{P}_x(X_t - x \in B)}{t}$$

Theorem

ν is a Lévy measure $\iff \displaystyle\int \min(1, |y|^2)\nu(dy) < \infty$ □

Theorem (Lévy-Ito decomposition)

Every Lévy process is a sum of:

(plus compensation)

- pure-jump process (described by the Lévy measure)
- Brownian motion (up to an affine map)
- uniform motion □

Lévy processes and non-local PDEs (1)

Theorem (a version of the Lévy-Khintchine formula)

The function:

$$u(t, x) = \mathbf{E}_x f(X_t)$$

solves a **'heat' equation**:

$$\frac{\partial u}{\partial t}(t, x) = (-\mathcal{A})u(t, x), \quad u(0, x) = f(x)$$

for a **pseudo-differential operator**:

$$(-\mathcal{A})f(x) = a f''(x) + b f'(x) + \int (f(x + y) - f(x) - f'(x)y\mathbf{1}_{|y|<1})\nu(dy) \quad \square$$

- $a \geq 0$: $d \times d$ matrix, 'Brownian part'
- $b \in \mathbf{R}^d$: 'drift'
- ν: Lévy measure, 'jump part'

Lévy processes and non-local PDEs (2)

Theorem

The function:

$$u(t,x) = \mathbf{E}_x(f(X_t); t < \tau_D)$$

solves the 'heat' equation in D with **exterior** condition:

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= (-\mathcal{A})u(t,x) && (t \ge 0,\ x \in D) \\ u(t,x) &= 0 && (t \ge 0,\ x \in D^c) \\ u(0,x) &= f(x) && (x \in D) \end{aligned}$$

□

- $\mathcal{A}$ is **non-local**!
- Hence $\mathcal{A}f(x)$ requires f to be defined everywhere (not just in a neighbourhood of x)
- $X(\tau_D) \in D^c$ instead of $X(\tau_D) \in \partial D$

Transition operators

Definition

We define **free transition kernel**:

$$p_t(x, A) = \mathbf{P}_x(X_t \in A)$$

and **transition kernel on *D***:

$$p_t^D(x, A) = \mathbf{P}_x(X_t \in A; t < \tau_D)$$

Definition

We define **free transition operators**:

$$P_t f(x) = \mathbf{E}_x f(X_t) = \int f(y) p_t(x, dy)$$

and **transition operators on *D***:

$$P_t^D f(x) = \mathbf{E}_x(f(X_t); t < \tau_D) = \int_D f(y) p_t^D(x, dy)$$

Generators

- $P_t P_s f = P_{t+s} f$
- $P_t^D P_s^D f = P_{t+s}^D f$

Definition

- $(-\mathcal{A})f = \lim_{t \to 0^+} \dfrac{P_t f - f}{t}$ (as in the 'heat' equation)
- $(-\mathcal{A}_D)f = \lim_{t \to 0^+} \dfrac{P_t^D f - f}{t}$

- When $f(x) = 0$ in D^c, then $\mathcal{A}_D f(x) = \mathcal{A} f(x)$
- $\mathcal{A}$ and $\mathcal{A}_D$ have different **domains**
- if $f \in \mathrm{Dom}(\mathcal{A}_D)$, then $f(x) = 0$ in D^c
- $\mathcal{A}$ and $\mathcal{A}_D$ are positive definite

Lévy-Khintchine formula

Definition

We define the **Lévy-Khintchine exponent**:

$$\Psi(\xi) = a\xi^2 + ib\xi + \int (1 - e^{-i\xi y} - i\xi y \mathbf{1}_{|y|<1})\nu(dy)$$

Theorem (Lévy-Khintchine formula)

$$\mathbf{E}_0 e^{-i\xi X_t} = e^{-t\Psi(\xi)}$$

$$\widehat{P_t f}(\xi) = e^{-t\Psi(\xi)}\hat{f}(\xi)$$

$$\widehat{\mathcal{A}f}(\xi) = \Psi(\xi)\hat{f}(\xi)$$

□

- $\Psi(\xi)$ is our **initial data**:
  all results will be given in terms of $\Psi(\xi)$

Kernels and densities

- Often $p_t(x, dy)$ and $p_t^D(x, dy)$ are absolutely continuous
- Also the Lévy measure $\nu(dy)$ will typically be absolutely continuous

Notation

For an absolutely continuous measure $\mu(dy)$, $\mu(y)$ denotes its density function

- $p_t(x, y)$ and $p_t^D(x, y)$ (if they exist) are called **transition densities** or **heat kernels**
- $p_t(x, y)$ depends only on $y - x$:

$$p_t(x, y) = p_t(y - x)$$

Summary

- We are given a Lévy-Khintchine exponent $\Psi(\xi)$ (think: $\Psi(\xi) = |\xi|^\alpha$, $0 < \alpha \le 2$)
- There is a corresponding pseudo-differential operator $\mathcal{A}$ and:
$$\widehat{\mathcal{A}f}(\xi) = \Psi(\xi)\hat{f}(\xi)$$
(think: $\mathcal{A} = (-\Delta)^{\alpha/2}$)
- Given a domain D, $\mathcal{A}_D$ is the operator $\mathcal{A}$ on D with 'Dirichlet' exterior condition
- We study **eigenvalues and eigenfunctions** of $\mathcal{A}_D$

- There is a Lévy process X_t corresponding to $\mathcal{A}$
- $\mathcal{A}_D$ corresponds to the process X_t killed at τ_D (the first exit time from D)

Half-line

Goal

Study the spectral theory for $\mathcal{A}_D$ and P_t^D for the **half-line**:

$$D = (0, \infty) \subseteq \mathbf{R}$$

- Why half-line?
  - explicit formulae (Part II)
  - applications in fluctuation theory (Part III)
  - model case for intervals and smooth domains in $\mathbf{R}^d$ (Part III)
  - possible applications in relativistic quantum physics
- The details are very technical, but the idea is simple
- We begin with two examples for which also the details are simple

Part I

Section 3

Warm-up: Brownian motion in an interval

BM in interval: The simplest example

- Let $D = (0, \pi)$ be the **interval**
- Let X_t be the **Brownian motion** with $\operatorname{Var} X_t = 2t$
- Then:
  - $\Psi(\xi) = \xi^2$
  - $(-\mathcal{A})f = \Delta f = f''$
  - $(-\mathcal{A}_D) = \Delta_D$ is the Dirichlet Laplacian
- Goal: eigenvalues and eigenfunctions of $\mathcal{A}_D$ and P_t^D

BM in interval: eigenvalues and eigenfunctions

- Eigenfunctions of $\mathcal{A}$ are sines and cosines
- Eigenfunctions and eigenvalues of $\mathcal{A}_D$ are

$$\begin{cases} f_n(x) = \sin(nx)\mathbf{1}_{x \in D} \\ \mu_n = n^2 \end{cases} \qquad n = 1, 2, \ldots$$

Indeed:

- $f_n''(x) = -n^2 f_n(x)$ in D
- $f_n(x) = 0$ in D^c
- f_n is continuous

BM in interval: solution

Solution

For:

$$f_n(x) = \sin(nx)\mathbf{1}_{x \in D}, \qquad \mu_n = n^2$$

we have:

$$P_t^D f_n = e^{-\mu_n t} f_n, \qquad \mathcal{A}_D f_n = \mu_n f_n$$

- Similar explicit solutions exist for balls, cubes etc.
- f_n form a complete orthogonal set in $L^2(D)$
- $\|f_n\|_2 = \sqrt{\frac{\pi}{2}}$
- $\frac{2}{\pi}\langle f, f_n \rangle$ is the Fourier series coefficient of f

BM in interval: eigenfunction expansion

Corollary

$$P_t^D f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\mu_n t} \langle f, f_n \rangle f_n(x)$$

$$p_t^D(x,y) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\mu_n t} f_n(x) f_n(y)$$

- There are better formulae for $p_t^D(x,y)$ for small t
- Results extend to:
  - more general processes
  - in general bounded domains

  But in general, there are no explicit formulae for μ_n and f_n!

Part I

Section 4

Motivation: Brownian motion in half-line

BM in half-line: The unbounded example

- Let $D = (0, \infty)$ be the **half-line**
- Let X_t be again the **Brownian motion**, $\operatorname{Var} X_t = 2t$:
  - $\Psi(\xi) = \xi^2$
  - $(-\mathcal{A})f = \Delta f = f''$
  - $(-\mathcal{A}_D) = \Delta_D$ is the Dirichlet Laplacian
- Goal: eigenvalues and eigenfunctions of $\mathcal{A}_D$ and P_t^D
- Main change: P_t^D are no longer **compact** operators

BM in half-line: eigenvalues and eigenfunctions

- Again, eigenfunctions and eigenvalues of $\mathcal{A}_D$ are:
$$\begin{cases} F_\lambda(x) = \sin(\lambda x)\mathbf{1}_{x>0} \\ \mu_\lambda = \lambda^2 \end{cases} \qquad \lambda \in (0, \infty)$$
  Indeed:
  - $F_\lambda''(x) = -\lambda^2 F_\lambda(x)$ in D
  - $F_\lambda(x) = 0$ in D^c
  - F_λ is continuous
- This time $F_\lambda \notin L^2(D)$!
- Note that $\mu_\lambda = \Psi(\lambda)$

BM in half-line: solution

Solution

For:

$$F_\lambda(x) = \sin(\lambda x)\mathbf{1}_{x>0}, \qquad \mu_\lambda = \lambda^2 = \Psi(\lambda)$$

we have:

$$P_t^D F_\lambda = e^{-t\,\Psi(\lambda)} F_\lambda, \qquad \mathcal{A}_D F_\lambda = \Psi(\lambda) F_\lambda$$

- F_λ are not in $L^2(D)$
- There are uncountably many eigenfunctions
- The Fourier sine transform of f is given by:

$$\langle f, F_\lambda \rangle = \int f(y) F_\lambda(y) dy$$

BM in half-line: eigenfunction expansion

Corollary

$$P_t^D f(x) = \frac{2}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} \langle f, F_\lambda \rangle F_\lambda(x) d\lambda$$

$$p_t^D(x,y) = \frac{2}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} F_\lambda(x) F_\lambda(y) d\lambda$$

- Here $f \in L^1(D)$
  Can be extended to $f \in L^2(D)$
- By **reflection principle**, there is a better formula:
  $$p_t^D(x,y) = p_t(y-x) - p_t(y+x) \quad (x, y \in D)$$
  No reflection principle for jump-type processes
- No similar results for other Lévy processes!
  (until very recently)

Problems:

(1) Let X_t be the Brownian motion and $D = (0, \infty)$. Prove, by a direct calculation, that $F_\lambda(x) = \sin(\lambda x)$ is the eigenfunction of P_t^D.
(Hint: use $p_t^D(x,y) = p_t(y-x) - p_t(y+x)$)

(2) Let X_t be the Brownian motion and $D = (0, \pi)$. Prove that for $x, y \in D$:

$$p_t^D(x,y) = \sum_{n=-\infty}^{\infty} p_t(y - x + 2n\pi) - \sum_{n=-\infty}^{\infty} p_t(y + x + 2n\pi)$$

(3) Show that for $D = (0, \infty)$, $\mathcal{A}_D$ and P_t^D may fail to be normal operators when X_t is not symmetric.

(4) Let X_t be the Brownian motion with drift $2b \in \mathbf{R}$ (that is, $\operatorname{Var} X_t = 2t$, $\mathbf{E}_x X_t = 2bt$) and $D = (0, \infty)$. Prove that $F_\lambda(x) = e^{-bx} \sin(\lambda x) \mathbf{1}_{x>0}$ ($\lambda > 0$) satisfies $\mathcal{A}_D F_\lambda(x) = (\lambda^2 + b^2) F_\lambda$ and $P_t^D F_\lambda = e^{-t(\lambda^2+b^2)} F_\lambda$. Give eigenfunction expansion for P_t^D on the weighted $L^2(D; e^{bx} dx)$ space. Find a (well-known) closed-form formula for $p_t^D(x,y)$.

Part I

Section 5

Complete monotonicity, complete Bernstein functions and subordinate Brownian motions

Assumptions

Goal

For a class of Lévy processes in half-line $D = (0, \infty)$:

- find a formula for eigenfunctions of $\mathcal{A}_D$ and P_t^D
- prove eigenfunctions expansion

- X_t should be symmetric (= isotropic)
  (otherwise, $\mathcal{A}_D$, P_t^D are not self-adjoint)
- some regularity is needed

Assumption (⚓)

- There is **no drift**
- The Lévy measure ν is:
  - **symmetric**
  - has **completely monotone** density on $(0, \infty)$

Complete monotonicity

Definition

$g(y)$ is **completely monotone** if:

$$(-1)^n g^{(n)}(y) \geq 0 \quad (n = 0, 1, 2, \ldots,\ y > 0)$$

Theorem (Sergei Natanovich Bernstein, 1929)

Equivalently: g is the **Laplace transform** of a measure:

$$g(y) = \mathcal{L}m(y) = \int_0^\infty e^{-sy} m(ds)$$

Proof

- ($\Leftarrow$) direct differentiation (easy)
- ($\Rightarrow$) inversion of Laplace transform (hard) $\square$

Complete Bernstein functions

Definition

$\Phi(\xi)$ is a **complete Bernstein function** (**CBF**) if:

- $\Phi : \mathbf{C} \setminus (-\infty, 0] \to \mathbf{C} \setminus (-\infty, 0]$ is **holomorphic**
- $\operatorname{Im} \Phi(\xi) \geq 0$ when $\operatorname{Im} \xi \geq 0$
- $\operatorname{Im} \Phi(\xi) \leq 0$ when $\operatorname{Im} \xi \leq 0$

- Equivalent definitions and properties of CBFs will be discussed later
- CBF $\equiv$ operator monotone function $\cong$ Pick function

Theorem

Assumption (⚓) $\iff$ $\Psi(\xi) = \Phi(\xi^2)$ for a **CBF** $\Phi(\xi)$ $\square$

- $\Psi(\xi)$ is the Lévy-Khintchine exponent of X_t
- Proof will be given later in this part

Subordinators

Definition

A **subordinator** is a nonnegative Lévy process (starting at 0)

Definition

We define the **Laplace exponent**:

$$\Phi(\xi) = b\xi + \int_0^\infty (1 - e^{-\xi y})\nu(dy)$$

$(b \geq 0)$

$(\nu((-\infty, 0)) = 0)$

$(\int_0^\infty \min(1, y)\nu(dy) < \infty)$

Theorem (Lévy-Khintchine formula for subordinators)

$$\mathbf{E}_0 e^{-\xi X_t} = e^{-t\Phi(\xi)}$$

$$\mathcal{L}(P_t f)(\xi) = e^{-t\Phi(\xi)}\mathcal{L}f(\xi)$$

$$\mathcal{L}(\mathcal{A}f)(\xi) = \Phi(\xi)\mathcal{L}f(\xi)$$

□

Subordinators and CBFs (1)

Theorem

(of a subordinator)

Laplace exponent $\Phi(\xi)$ is a CBF

$\Updownarrow$

(of a subordinator)

Lévy measure ν has completely monotone density

Proof

- Suppose that $\nu(y) = \mathcal{L}m(y)$ (that is, $\nu(dy) = (\mathcal{L}m(y))dy$)
- $$\begin{aligned}\Phi(\xi) &= b\xi + \int_0^\infty (1 - e^{-\xi y})\nu(dy) \\ &= b\xi + \int_0^\infty \int_0^\infty (1 - e^{-\xi y})e^{-sy} m(ds)dy \\ &= b\xi + \int_0^\infty \frac{\xi}{s+\xi} \frac{m(ds)}{s}\end{aligned}$$

Subordinators and CBFs (2)

Proof (cont.)

- $\Phi(\xi) = b\xi + \int_0^\infty \frac{\xi}{s+\xi} \frac{m(ds)}{s}$ *(more precisely: extends to a CBF)*
- By checking $\operatorname{Im}\Phi(\xi)$, $\Phi(\xi)$ is a CBF
- Reasoning can be reversed by the next result. □

Theorem

$\Phi(\xi)$ is a CBF $\iff$ $\Phi(\xi) = a + b\xi + \int_0^\infty \frac{\xi}{s+\xi} \frac{m(ds)}{s}$ $(a, b \geq 0)$

$(\int_0^\infty \min(1, y) m(dy) < \infty)$

Proof

- ($\Leftarrow$) direct calculation (easy)
- ($\Rightarrow$) representation of positive harmonic functions by the Poisson kernel (harder) □

Subordination

Definition

- If Y_t is a stochastic process, Z_t is a subordinator, and Y_t, Z_t are independent processes, then $X_t = Y(Z_t)$ is a **subordinate process**
- If Y_t is Brownian motion, then X_t is **subordinate Brownian motion**

Theorem

Suppose that:

- Y_t is Brownian motion ($\operatorname{Var} Y_t = 2t$, $\Psi_Y(\xi) = \xi^2$)
- Z_t is a subordinator
- $\Phi_Z(\xi)$ is the Laplace exponent of Z_t

Then $\Psi_X(\xi) = \Phi_Z(\xi^2)$ is the Lévy-Khintchine exp. of X_t

Proof

- Direct calculation: a nice exercise

Summary

Theorem (equivalent forms of Assumption (⚓))

- X_t has no drift
- The Lévy measure ν of X_t is:
  - symmetric
  - has completely monotone density on $(0, \infty)$

$$\Updownarrow$$

- The Lévy-Khintchine exponent of X_t satisfies $\Psi(\xi) = \Phi(\xi^2)$ for a CBF $\Phi(\xi)$

$$\Updownarrow$$

- $X_t = Y(Z_t)$ is a subordinate Brownian motion
- The Lévy measure ν_Z of Z_t has completely monotone density on $(0, \infty)$

Examples

Process:	**Stable**	**Relativistic**	**Var. gamma**
Parameter:	$\alpha \in (0, 2)$	$m \in (0, \infty)$	—
$\Psi(\xi)$	$\lvert\xi\rvert^\alpha$	$\sqrt{\xi^2 + m^2} - m$	$\log(\xi^2 + 1)$
$\nu_X(y)$	$\dfrac{c_\alpha}{\lvert y\rvert^{1+\alpha}}$	$\dfrac{m\,K_1(m\lvert y\rvert)}{\pi\,\lvert y\rvert}$	$\dfrac{e^{-\lvert y\rvert}}{\lvert y\rvert}$
$\Phi(\xi)$	$\xi^{\alpha/2}$	$\sqrt{\xi + m^2} - m$	$\log(\xi + 1)$
$\nu_Z(s)$	$\dfrac{c_\alpha}{s^{1+\alpha/2}}$	$\dfrac{e^{-ms}}{2\sqrt{\pi}\,s^{3/2}}$	$\dfrac{e^{-s}}{s}$

(K_1 is a Bessel function)

Proof ((3) ⇒ (1))

- Suppose that $X_t = Y(Z_t)$, Y_t is BM, Z_t is a subordinator and $\nu_Z(s) = \mathcal{L}m(s)$ *(X_t symmetric ⇒ no drift)*
- $p_{X,t}(y) = \int_0^\infty p_{Y,s}(y) p_{Z,t}(s) ds$ (subordination formula)
- $\nu_X(y) = \lim_{t \to 0^+} \frac{p_{X,t}(y)}{t} = \int_0^\infty p_{Y,s}(y) \nu_Z(s) ds$
- By $p_{Y,s}(y) = \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{y^2}{4s}\right)$, $\nu_Z(s) = \int_0^\infty e^{-st} m(dt)$:

$$\nu_X(y) = \int_0^\infty \left(\int_0^\infty \frac{1}{\sqrt{4\pi s}} \exp\left(-\frac{y^2}{4s}\right) e^{-st} ds \right) m(dt)$$

$$= \int_0^\infty \frac{1}{2\sqrt{t}} e^{-\sqrt{t}|y|} m(dt) = \mathcal{L}\tilde{m}(|y|) \qquad \square$$

Proof ((1) ⇒ (3))

- Reverse the reasoning □

Proof ((2) ⟺ (3))

- $\Psi_X(\xi) = \Phi_Z(\xi^2) \iff X_t = Y(Z_t)$ (a theorem above)
- $\Phi_Z(\xi)$ is a CBF $\iff \nu_Z(s)$ is completely monotone (another theorem above) □

Rene Schilling, Renming Song, Zoran Vondraček
Bernstein Functions: Theory and Applications
De Gruyter, 2010

Problems:

(1) Let $\Psi_Y(\xi)$ be the Lévy-Khintchine exponent of a Lévy process Y_t, and $\Phi_Z(\xi)$ be the Laplace exponent of a subordinator Z_t. Suppose that Y_t and Z_t are independent processes. Show that the Lévy-Khintchine exponent of $X_t = Y(Z_t)$ is $\Psi_X(\xi) = \Phi_Z(\Psi_Y(\xi))$.

(Note: $\operatorname{Re} \Psi_Y(\xi) \geq 0$ and $\Phi_Z(\xi)$ is well-defined if $\operatorname{Re} \xi \geq 0$)

(2) Prove that $\Phi(\xi)$ is a Laplace exponent (a.k.a. **Bernstein function**) if and only if $\Phi(0) \geq 0$ and $\Phi'(\xi)$ is completely monotone.

(3) Suppose that $\Phi(\xi)$, $\Phi_1(\xi)$, $\Phi_2(\xi)$ are non-zero CBFs and $c > 0$, $0 < \alpha < 1$. Prove that:

(a) $c\Phi(\xi), \quad \Phi_1(\xi) + \Phi_2(\xi), \quad \Phi_1(\Phi_2(\xi)), \quad \dfrac{\xi}{\Phi(\xi)}, \quad (\Phi_1(\xi))^\alpha (\Phi_2(\xi))^{1-\alpha}$ are CBFs;

(b) $\xi^{1-\alpha}\Phi(\xi^\alpha)$ is a CBF;

(Hint: use only '$\Phi(\xi)$ is CBF $\Rightarrow$ $\Phi(\xi^\alpha)$ is CBF' and '$\Phi(\xi)$ is CBF $\Rightarrow$ $\xi/\Phi(\xi)$ is CBF')

(c) $(\Phi(\xi^\alpha))^{1/\alpha}$ is a CBF;

(d) Φ maps $\{\xi \in \mathbf{C} : \operatorname{Arg} \xi \in (0, \alpha\pi)\}$ into itself;

(e) $(\Phi_1(\xi^\alpha) + \Phi_2(\xi^\alpha))^{1/\alpha}, \quad ((\Phi_1(\xi))^\alpha + (\Phi_2(\xi))^\alpha)^{1/\alpha}, \quad \Phi_1(\xi^\alpha)\Phi_2(\xi^{1-\alpha})$ are CBFs.

Part II

Eigenfunction expansion in half-line

- Formula for eigenfunctions
- Properties of eigenfunctions
- Eigenfunction expansion

Note: There are a lot of ugly formulae in this part!

Part II

Section 1

Formula for eigenfunctions

Setting

Assumptions

Throughout this part we assume that:

- X_t is a (symmetric) Lévy process in $\mathbf{R}$
- $\Psi(\xi)$ is the Lévy-Khintchine exponent of X_t
- Assumption (⚓):

$$\Psi(\xi) = \Phi(\xi^2) \text{ for a } \textbf{CBF } \Phi(\xi)$$

- P_t are free transition operators of X_t
  $\mathcal{A}$ is the generator of P_t
- $D = (0, \infty)$
- P_t^D are transition operators of X_t on D
  $\mathcal{A}_D$ is the generator of P_t^D

Eigenfunctions: intuition

- For $F(x) = \sin(\lambda x + \theta)$: $(\widehat{F} = c\delta_\lambda - \bar{c}\delta_\lambda \textit{ for } c = -\frac{i}{2}e^{i\theta})$
$$\mathcal{A}F = \Psi(\lambda)F, \qquad P_t F = e^{-t\Psi(\lambda)}F$$
(Lévy-Khintchine formula, $\Psi(\lambda) = \Psi(-\lambda)$)
- For $g(x) = f(x)\mathbf{1}_{x>0}$ and $x \in D$ large:
$$\mathcal{A}_D g(x) = \mathcal{A}g(x) \approx \mathcal{A}f(x)$$

Guess

For each $\lambda > 0$ there is $F_\lambda(x)$ such that:
$$\mathcal{A}_D F_\lambda = \Psi(\lambda)F_\lambda, \qquad P_t^D F_\lambda = e^{-t\Psi(\lambda)}F_\lambda$$
and $F_\lambda(x) \approx \sin(\lambda x + \theta_\lambda)$ as $x \to \infty$. That is:
$$F_\lambda(x) = \sin(\lambda x + \theta_\lambda)\mathbf{1}_{x>0} - G_\lambda(x)$$
where G_λ is **small**.

- In this part, always $x > 0$
- For simplicity, we drop $\mathbf{1}_{x>0}$ from the notation

Eigenfunctions: formula (1)

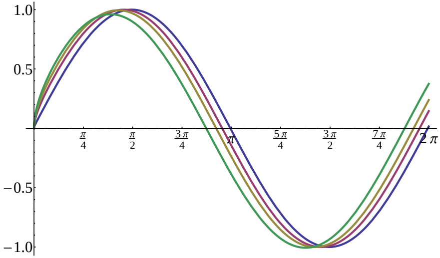
Theorem [K, 2010]

For each $\lambda > 0$ there are:

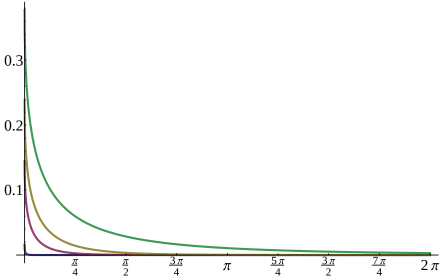
- $\theta_\lambda \in [0, \pi/2)$
- completely monotone $G_\lambda(x)$

such that $F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - G_\lambda(x)$ satisfies:

$$\mathcal{A}_D F_\lambda = \Psi(\lambda) F_\lambda, \quad P_t^D F_\lambda = e^{-t\Psi(\lambda)} F_\lambda \qquad \square$$



$F_\lambda(x/\lambda)$ for the relativistic process ($\lambda = \frac{1}{20}, \frac{1}{2}, 1, 10$)



corresponding $G_\lambda(x/\lambda)$

Eigenfunctions: formula (2)

- $F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - G_\lambda(x)$
- G_λ is completely monotone: $G_\lambda(x) = \mathcal{L}\gamma_\lambda(x)$

Theorem [K, 2010]

$$\theta_\lambda = \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 - u^2} \log \frac{2\lambda(\Psi(\lambda) - \Psi(u))}{\Psi'(\lambda)(\lambda^2 - u^2)} du$$

$$\gamma_\lambda(ds) = \frac{1}{2\pi} \left(\operatorname{Im} \frac{\Psi'(\lambda)}{\Psi(\lambda) - \Phi^+(-s^2)} \right) \times \exp\left(\frac{1}{\pi} \int_0^\infty \frac{s}{s^2 + u^2} \log \frac{2\lambda(\Psi(\lambda) - \Psi(u))}{\Psi'(\lambda)(\lambda^2 - u^2)} du \right) ds \quad \square$$

- γ_λ may fail to have density!
- $\Phi^+(-s^2) = \lim_{\varepsilon \to 0^+} \Phi(-s^2 + \varepsilon i)$ in the distributional sense

Eigenfunctions: stable processes

Example

(symmetric)

For the **α-stable process**, $\Psi(\xi) = |\xi|^\alpha$, $\alpha \in (0, 2)$:

$$F_\lambda(x) = \sin\left(\lambda x + \frac{(2-\alpha)\pi}{8}\right) - \int_0^\infty \gamma(s) e^{-\lambda s x} ds$$

$$\gamma(s) = \frac{\sqrt{2\alpha}\,\sin(\alpha\pi/2)}{2\pi} \frac{s^\alpha}{1 + s^{2\alpha} - 2s^\alpha \cos(\alpha\pi/2)} \times \exp\left(\frac{1}{\pi}\int_0^\infty \frac{1}{1+u^2} \log \frac{1 - s^2u^2}{1 - s^\alpha u^\alpha} du\right)$$

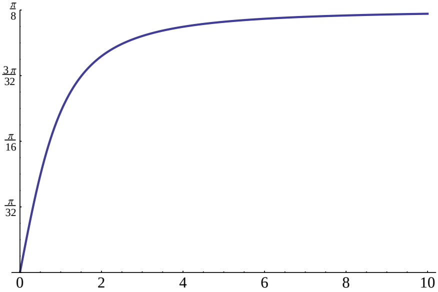
- Scaling: $F_\lambda(x) = F_1(\lambda x)$
- $F_\lambda(x) \sim \dfrac{1}{\sqrt{\alpha/2}\,\Gamma(\alpha/2)} (\lambda x)^{\alpha/2}$ as $x \to 0^+$

Eigenfunctions: relativistic processes

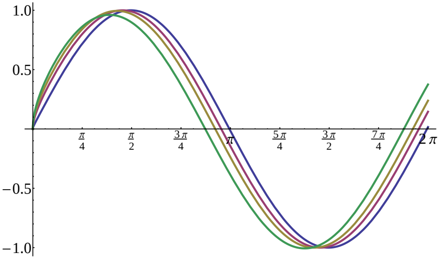
Example

For the **relativistic process**, $\Psi(\xi) = \sqrt{\xi^2 + m^2} - m$:

- θ_λ increases from 0 to $\pi/8$
- $F_\lambda(x) \sim \sqrt{\dfrac{2\lambda x}{\pi}}$ as $x \to 0^+$



θ_λ



$F_\lambda(x/\lambda)$, $\lambda = \frac{1}{20}, \frac{1}{2}, 1, 10$

Eigenfunctions: two more examples

Example

For the **variance gamma process**, $\Psi(\xi) = \log(\xi^2 + 1)$:

- θ_λ increases from 0 to $\pi/4$
- $F_\lambda(x) \sim \dfrac{\lambda}{\sqrt{2}\,\sqrt{\lambda^2+1}} \dfrac{1}{\sqrt{|\log x|}}$ as $x \to 0^+$

Example

For the **mixture of stables**, $\Psi(\xi) = \xi^\alpha + \xi^\beta$,
(sum of two independent stables, $0 < \alpha \le \beta \le 2$):

- θ_λ decreases from $\dfrac{(2-\alpha)\pi}{8}$ to $\dfrac{(2-\beta)\pi}{8}$
- $F_\lambda(x) \sim \dfrac{1}{\sqrt{\beta/2}\,\Gamma(\beta/2)} (\lambda x)^{\beta/2}$ as $x \to 0^+$

Eigenfunctions: yet another two examples

Example

For the Brownian motion, $\Psi(\xi) = \xi^2$:

- $\log \dfrac{2\lambda(\Psi(\lambda) - \Psi(u))}{\Psi'(\lambda)(\lambda^2 - u^2)} = 0$
- $\theta_\lambda = 0$, $\gamma_\lambda = 0$ and $F_\lambda(x) = \sin(\lambda x)$, as expected

Example

For $\Psi(\xi) = \dfrac{\xi}{1 + \xi}$, $\nu(y) = \dfrac{e^{-|y|}}{2}$ (compound Poisson with Laplace distributed jumps):

- $\theta_\lambda = \arctan \lambda$ increases from 0 to $\pi/2$
- γ_λ vanishes!
- $F_\lambda(x) = \sin(\lambda x + \arctan \lambda) \mathbf{1}_{x>0}$
- F_λ is discontinuous at 0!

Part II

Section 2

Properties of eigenfunctions

Laplace transform of eigenfunctions

- Derivation of the formula for F_λ will be sketched in Part IV
- **Bounds** and **asymptotics** of F_λ can be proved in a fairly general setting
- Most of them follow from the formula for $\mathcal{L}F_\lambda$
- In most applications, exact formula is not needed

Theorem [K, 2010]

$$\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2 + s^2} \times \exp\left(\frac{1}{\pi}\int_0^\infty \frac{s}{s^2+u^2} \log \frac{\Psi'(\lambda)(\lambda^2-u^2)}{2\lambda(\Psi(\lambda)-\Psi(u))}\, du\right) \quad \square$$

Common elements (the worst slide ever!)

$$\theta_\lambda = \frac{1}{\pi}\int_0^\infty \frac{\lambda}{\lambda^2 - u^2} \log \frac{2\lambda(\psi(\lambda) - \psi(u))}{\psi'(\lambda)(\lambda^2 - u^2)}\, du$$

$$\gamma_\lambda(ds) = \frac{1}{2\pi}\left(\operatorname{Im} \frac{\psi'(\lambda)}{\psi(\lambda) - \phi^+(-s^2)}\right) \times \exp\left(\frac{1}{\pi}\int_0^\infty \frac{s}{s^2 + u^2} \log \frac{2\lambda(\psi(\lambda) - \psi(u))}{\psi'(\lambda)(\lambda^2 - u^2)}\, du\right) ds$$

$$\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2 + s^2} \exp\left(\frac{1}{\pi}\int_0^\infty \frac{s}{s^2 + u^2} \log \frac{\psi'(\lambda)(\lambda^2 - u^2)}{2\lambda(\psi(\lambda) - \psi(u))}\, du\right)$$

Definition

$$\phi_\lambda^\dagger(\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + u^2} \log \frac{\psi'(\lambda)(\lambda^2 - u^2)}{2\lambda(\psi(\lambda) - \psi(u))}\, du\right)$$

Simplification

- $\Psi(\xi) = \Phi(\xi^2)$
- We will now use mostly $\Phi(\xi)$, not $\Psi(\xi)$

Definition

- $\Phi_\lambda(\xi^2) = \dfrac{\Phi'(\lambda^2)(\lambda^2 - \xi^2)}{\Phi(\lambda^2) - \Phi(\xi^2)} = \dfrac{\Psi'(\lambda)(\lambda^2 - \xi^2)}{2\lambda(\Psi(\lambda) - \Psi(\xi))}$
- $\Phi^\dagger(\xi) = \exp\left(\dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{\xi}{\xi^2 + u^2} \log \Phi(u^2) du\right)$
- $\Phi^\dagger_\lambda = (\Phi_\lambda)^\dagger$

- $\operatorname{Arg} \Phi^\dagger(i\xi) = -\dfrac{1}{\pi}\displaystyle\int_0^\infty \dfrac{\xi}{\xi^2 - u^2} \log \Phi(u^2) du$

Eigenfunctions revisited

Theorem

We have:

$$F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - \mathcal{L}\gamma_\lambda(x)$$

with:

$$\theta_\lambda = \operatorname{Arg}(\Phi_\lambda^\dagger(i\lambda))$$

$$\gamma_\lambda(ds) = \frac{1}{\pi}\,\frac{\lambda}{\lambda^2+s^2}\,\frac{\operatorname{Im}(\Phi_\lambda)^+(-s)}{\Phi_\lambda^\dagger(s)}\,ds$$

$$\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2+s^2}\,\Phi_\lambda^\dagger(s)$$

- Technical details are now moved to definitions

Divide and rule

Strategy [K, 2010], [K-Małecki-Ryznar, 2011]

(1) study Φ_λ
(2) study $\Phi^\dagger$
(3) use (1) and (2) to θ_λ
(4) apply (1) and (2) to $\mathcal{L}F_\lambda$
(5) use tauberian theory (Jovan Karamata et al.)
(properties of $\mathcal{L}F_\lambda \Rightarrow$ properties of F_λ)

- Alternatively, for a specific Ψ, one may try:

(4') apply (1) and (2) to γ_λ
(5') use abelian theory
(properties of $\gamma_\lambda \Rightarrow$ properties of $\mathcal{L}\gamma_\lambda$)

Properties of Φ_λ

$$\Phi_\lambda(\xi) = \frac{\Phi'(\lambda^2)(\lambda^2 - \xi)}{\Phi(\lambda^2) - \Phi(\xi)}$$

Lemma

$\Phi(\xi)$ is a CBF $\Rightarrow \Phi_\lambda(\xi)$ is a CBF $\square$

- Proof: nice exercise
- estimates of Φ_λ depend on bounds on $\dfrac{-\xi\Phi''(\xi)}{\Phi'(\xi)}$
- $\Phi_\lambda(\xi^2) \sim \Phi'(\lambda^2)\dfrac{\xi^2}{\Phi(\xi^2)}$ as $\xi \to \infty$
- $\Phi_\lambda(\xi^2) \to \Phi'(\lambda^2)\dfrac{\lambda^2}{\Phi(\lambda^2)}$ as $\xi \to 0^+$

Properties of $\Phi^\dagger$ (1)

$$\Phi^\dagger(\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2+u^2}\log\Phi(u^2)du\right)$$

Lemma [K, 2010], [Kim-Song-Vondraček, 2010], [Rogers, 1983]
(and discoverers of the Wiener-Hopf factorization for Lévy processes)

- $\Phi(\xi)$ is a CBF $\Rightarrow \Phi^\dagger(\xi)$ is a CBF
- $\Phi^\dagger(\xi)\Phi^\dagger(-\xi) = \Phi(-\xi^2)$ □

- Proof: smart contour integration
- A fundamental lemma!
- It enables inversion of the Laplace transform in
$$\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2+s^2}\Phi_\lambda^\dagger(s)$$
- Residues at $\pm i\lambda \mapsto \sin(\lambda x + \theta_\lambda)$
- Jump along $(-\infty, 0] \mapsto \mathcal{L}\gamma_\lambda(x)$

Properties of $\Phi^\dagger$ (2)

$$\Phi^\dagger(\xi) = \exp\left(\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2+u^2}\log\Phi(u^2)du\right)$$

Bounds [K-Małecki-Ryznar, 2011], [Kim-Song-Vondraček, 2010]

$$\tfrac{1}{2}\sqrt{\Phi(\xi^2)} \le \Phi^\dagger(\xi) \le 2\sqrt{\Phi(\xi^2)} \qquad \square$$

Proof

- $\min\left(1, \dfrac{u^2}{\xi^2}\right) \le \dfrac{\Phi(u^2)}{\Phi(\xi^2)} \le \max\left(1, \dfrac{u^2}{\xi^2}\right)$
- $\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{\xi}{\xi^2+u^2}\log\left(\max\left(1, \frac{u^2}{\xi^2}\right)\right)du \le \log 2$
- Similar estimate for the lower bound $\square$

Properties of $\Phi^\dagger$ (3)

Definition

$\Phi(\xi)$ is **regularly varying** of order α (α-**RV**) at 0^+ if

$$\lim_{\xi \to 0^+} \frac{\Phi(c\xi)}{\Phi(\xi)} = c^\alpha \qquad \text{for all } c > 0$$

$\Phi(\xi)$ is **regularly varying** of order α (α-**RV**) at ∞ if

$$\lim_{\xi \to \infty} \frac{\Phi(c\xi)}{\Phi(\xi)} = c^\alpha \qquad \text{for all } c > 0$$

Asymptotics [K-Małecki-Ryznar, 2011], [Kim-Song-Vondraček, 2010]

- $\Phi^\dagger(\xi) \sim \sqrt{\Phi(\xi^2)}$ as $\xi \to \infty$ if $\Phi(\xi)$ is RV at ∞
- $\Phi^\dagger(\xi) \sim \sqrt{\Phi(\xi^2)}$ as $\xi \to 0^+$ if $\Phi(\xi)$ is RV at 0^+ $\square$

- Proof: explicit estimates

Properties of θ_λ

$$\theta_\lambda = \operatorname{Arg} \Phi_\lambda^\dagger(i\lambda)$$

- $\theta_\lambda \in [0, \pi/2)$ $\quad (\Phi_\lambda^\dagger(\xi) \not\equiv c\,\xi$, so $\theta_\lambda \ne \pi/2)$
- θ_λ close to $\pi/2$ generates problems
- $\theta_\lambda \le \arctan \sqrt{\dfrac{\Phi(\lambda^2)}{\lambda^2 \Phi'(\lambda^2)} - 1}$
  (good for $\Phi(\xi)$ with power-type growth)

Bounds [K-Małecki-Ryznar, 2011]

$$\left(\inf_{\xi > 0} \frac{-\xi \Phi''(\xi)}{\Phi'(\xi)} \right) \frac{\pi}{4} \le \theta_\lambda \le \left(\sup_{\xi > 0} \frac{-\xi \Phi''(\xi)}{\Phi'(\xi)} \right) \frac{\pi}{4} \qquad \square$$

- Proof: bounds for Φ_λ, explicit formula for $\Phi(\xi) = \xi^{\alpha/2}$

Properties of F_λ (1)

$$\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2 + s^2}\,\Phi_\lambda^\dagger(s)$$

$$F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - \mathcal{L}\gamma_\lambda(x)$$

Bounds [K-Małecki-Ryznar, 2011]

When $\lambda x \le \frac{1}{2}(\frac{\pi}{2} - \theta_\lambda)$, then:

$$\frac{1}{5}\,\lambda x \sqrt{\Phi_\lambda(1/x^2)} \le F_\lambda(x) \le 30(\tfrac{\pi}{2} - \theta_\lambda)\,\lambda x\,\sqrt{\Phi_\lambda(1/x^2)} \qquad \square$$

- Proof: concavity of $F_\lambda(x)$ for small x, comparison of Laplace transforms
- Kind of uniform continuity of $F_\lambda(x/\lambda)$

Properties of F_λ (2)

$$\mathcal{L}F_\lambda(s) = \frac{\lambda}{\lambda^2 + s^2} \, \Phi_\lambda^\dagger(s)$$

$$F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - \mathcal{L}\gamma_\lambda(x)$$

Asymptotics [K, 2010], [K-Małecki-Ryznar, 2011]

- $F_\lambda(x) \sim \dfrac{\sqrt{\lambda^2 \Phi'(\lambda^2)}}{\Gamma(1+\alpha)} \dfrac{1}{\sqrt{\Phi(1/x^2)}}$ as $x \to 0^+$

  if $\Phi(\xi)$ is α-RV at ∞

- $F_\lambda(x) \sim V(x)\sqrt{\lambda^2 \Phi'(\lambda^2)}$ as $\lambda \to 0^+$

  if $\limsup_{\lambda \to 0^+} \theta_\lambda < \pi/2$ $\square$

- Proof: technical, nothing interesting
- $V(x)$ comes from fluctuation theory, $\mathcal{L}V(\xi) = \dfrac{1}{\xi \Phi^\dagger(\xi)}$

Part II

Section 3

Eigenfunction expansion

Eigenfunction expansion (1)

Guess

$$P_t^D f(x) = \frac{2}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} \langle f, F_\lambda \rangle F_\lambda(x) d\lambda$$

$$\mathcal{A}_D f(x) = \frac{2}{\pi} \int_0^\infty \Psi(\lambda) \langle f, F_\lambda \rangle F_\lambda(x) d\lambda$$

$$p_t^D(x, y) = \frac{2}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} F_\lambda(x) F_\lambda(y) d\lambda$$

- Some delicate problems with integrability arise when $e^{-t\Psi(\lambda)}$ is not integrable
- Solution: use continuous $L^2(D)$ extension

Eigenfunction expansion (2)

Definition

$$\Pi f(\lambda) = \int_0^\infty f(x) F_\lambda(x) dx = \langle f, F_\lambda \rangle$$

$$\Pi^* g(x) = \int_0^\infty g(\lambda) F_\lambda(x) d\lambda$$

Theorem [K, 2010], [K-Małecki-Ryznar, 2011]

- $\sqrt{\frac{2}{\pi}}\Pi$, $\sqrt{\frac{2}{\pi}}\Pi^*$ extend to **unitary** operators on $L^2(D)$
- $f \in \mathrm{Dom}_{L^2(D)}(\mathcal{A}_D) \iff \Psi(\lambda)\Pi f(\lambda) \in L^2(D)$
- $\Pi(\mathcal{A}_D f)(\lambda) = \Psi(\lambda)\Pi f(\lambda)$ for $f \in \mathrm{Dom}_{L^2(D)}(\mathcal{A}_D)$
- $\Pi(P_t^D f)(\lambda) = e^{-t\Psi(\lambda)}\Pi f(\lambda)$ for $f \in L^2(D)$ □

Eigenfunction expansion (3)

Corollary

If $f \in L^1(D)$ and $e^{-t\Psi(\lambda)}$ is integrable, then:

$$P_t^D f(x) = \frac{2}{\pi} \Pi^*(\Psi \cdot (\Pi f))(x) = \frac{2}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} \langle f, F_\lambda \rangle F_\lambda(x) d\lambda$$

If $e^{-t\Psi(\lambda)}$ is integrable, then:

$$p_t^D(x,y) = \frac{2}{\pi} \int_0^\infty e^{-t\Psi(\lambda)} F_\lambda(x) F_\lambda(y) d\lambda$$

- Proofs will be sketched in Part IV
- Most difficult part: completeness of F_λ

Problems:

(1) Show that if $\Phi(\xi)$ is a CBF, then $\Phi_\lambda(\xi)$ is a CBF.

(2) Prove, by a direct calculation, that for $\Psi(\xi^2+1)$ (that is, $\nu(y) = e^{-|y|}/2$), $F_\xi(x) = \sin(\xi x + \arctan \xi)\mathbf{1}_{\xi>0}$ is an eigenfunction in $(0, \infty)$.

(3) Prove that P_t^D has no $L^2(D)$ eigenfunctions when X_t is the symmetric α-stable process, $\Psi(\xi) = |\xi|^\alpha$.

Note: this is true in the general case under Assumption (⚓), but the proof is much more difficult.

(4) Show that P_t^D may have $L^2(D)$ eigenfunctions when X_t is not symmetric.

Open problems:

(1) Are there any other eigenfunctions F of P_t^D?

(2) Formula for $\mathcal{L}F_\lambda$ makes sense for a much more general class of Lévy-Khintchine exponents $\Psi(\xi)$. For which exponents does this formula indeed define the Laplace transform of a function?

(3) Is it true that $p_t^D(x,y) = \lim\limits_{\varepsilon\to 0^+} \dfrac{2}{\pi}\displaystyle\int_0^\infty e^{-\varepsilon\lambda - t\Psi(\lambda)} F_\lambda(x) F_\lambda(y) d\lambda$ when $e^{-t\Psi(\lambda)}$ is not integrable?

Part III

Applications

- Supremum functional and first passage times
- Connections with fluctuation theory
- Eigenvalues for intervals
- Higher-dimensional domains

Part III

Section 1

Supremum functional and first passage times

Supremum functional and FPT (1)

- In this section we often write $\mathbf{P}$ for $\mathbf{P}_0$ (that is, X_t starts at 0)

Definition

We define the the **first passage time** (**FPT**):

$$\tau_x = \inf\{s \geq 0 : X_s \geq x\}$$

and the **supremum functional** (or **sup. process**):

$$M_t = \sup_{s \in [0,t]} X_s$$

- Important in many areas of applied probability
- Distribution is rather difficult to compute

Supremum functional and FPT (2)

Proposition

$$\mathbf{P}(M_t < x) = \mathbf{P}(\tau_x > t)$$

Proof

- $\{M_t < x\}$ is **almost** equal to $\{\tau_x > t\}$
- Use càdlàg paths and quasi left continuity $\square$

- Let $D = (0, \infty)$
- When X_t is symmetric, then $\mathbf{P}(\tau_x > t) = \mathbf{P}_x(\tau_D > t)$
- (In the general case, $\mathbf{P}(\tau_x > t) = \mathbf{P}_{-x}(\tau_{(-\infty,0)} > t)$)
- $\mathbf{P}_x(\tau_D > t) = \int_0^\infty p_t^D(x, y) dy$
- We have a formula for $p_t^D(x, y)$

Formula for FPT (1)

Theorem [K, 2010], [K-Małecki-Ryznar, 2011]

Under Assumption (⚓), and if:

- $\sup_{\lambda>0} \theta_\lambda < \frac{\pi}{2}$
- $\sqrt{\frac{\Psi'(\lambda)}{2\lambda\Psi(\lambda)}} e^{-t\Psi(\lambda)}$ is integrable at ∞ (*note that* $\frac{\Psi'(\lambda)}{2\lambda\Psi(\lambda)} = \frac{\Phi'(\lambda^2)}{\Phi(\lambda^2)} \le \frac{1}{\lambda^2}$)

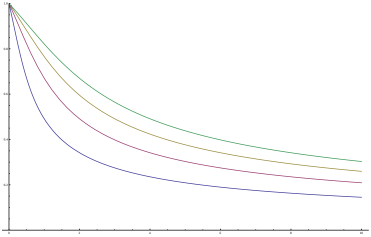
we have:

$$\mathbf{P}(\tau_x > t) = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{\Psi'(\lambda)}{2\lambda\Psi(\lambda)}} e^{-t\Psi(\lambda)} F_\lambda(x) d\lambda$$

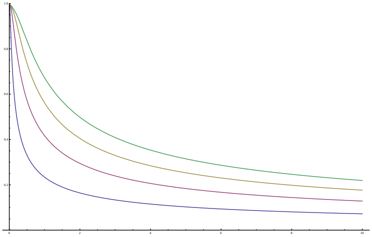
- Integrability near 0 is automatic!

Examples

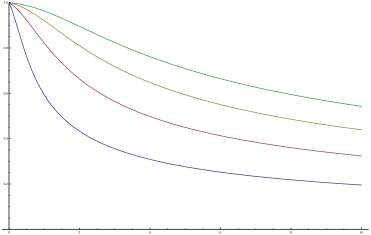
- Assumptions are relatively mild, examples include:
  - Symmetric stable processes
  - Relativistic processes
  - Variance gamma process
  - Mixtures of stables
  - $\Psi(\xi) = \log(\log(\xi^2 + 1) + 1)$ when $t \geq 1/2$
- Problems when $\Psi(\xi)$ grows very slowly, for example, for compound Poisson processes
- Formula is applicable for numerical computations (although there are essential problems with numerical stability)



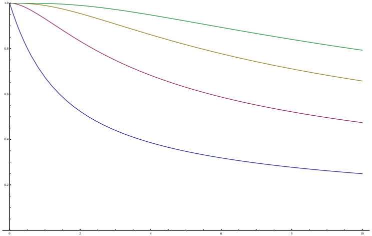
$\Psi(\xi) = |\xi|$



$\Psi(\xi) = |\xi| + \xi^2$



$\Psi(\xi) = \sqrt{\xi^2 + 1} - 1$



$\Psi(\xi) = 1 - \cos(\xi/2)$ *(here (⚓) fails!)*

(using Baxter and Donsker's result)

Plots of $\mathbf{P}(\tau_x > t)$ for $x = 0.5$, 1, 1.5 and 2

Formula for FPT (2)

Proof no. 1

- $\mathbf{P}(\tau_x > t) = \int_0^\infty p_t^D(x,y)dy = \lim_{\varepsilon\to 0^+} \int_0^\infty e^{-\varepsilon y} p_t^D(x,y)dy$
- $p_t^D(x,y) = \frac{2}{\pi}\int_0^\infty e^{-t\Psi(\lambda)} F_\lambda(x) F_\lambda(y) d\lambda$
- $\int_0^\infty e^{-\varepsilon y} F_\lambda(y) = \mathcal{L}F_\lambda(y)$
- $\mathbf{P}(\tau_x > t) = \frac{2}{\pi} \lim_{\varepsilon\to 0^+} \int_0^\infty e^{-t\Psi(\lambda)} F_\lambda(x) \mathcal{L}F_\lambda(\varepsilon) d\lambda$
- $\mathcal{L}F_\lambda(\varepsilon) \to \sqrt{\frac{\Psi'(\lambda)}{2\lambda\Psi(\lambda)}}$ as $\varepsilon \to 0^+$
- Show uniform integrability as $\varepsilon \to 0^+$ (very technical) $\square$

Formula for FPT (3)

Proof no. 2

- $$\int_0^\infty \int_0^\infty e^{-\xi x} e^{-zt} \mathbf{P}(\tau_x > t) dx\, dt$$ is known (Baxter-Donsker formula, discussed later)
- $$\int_0^\infty \int_0^\infty e^{-\xi x} e^{-zt} \left(\frac{2}{\pi} \int_0^\infty \sqrt{\frac{\psi'(\lambda)}{2\lambda\psi(\lambda)}}\, e^{-t\psi(\lambda)} F_\lambda(x) d\lambda \right) dx\, dt$$ can be computed (slightly less technical) *(compared to Proof no. 1)*
- Both turn out to be equal
- Use uniqueness argument for Laplace transform $\square$

Properties of FPT (1)

Corollary

The formula can be differentiated under the integral when $\sqrt{\frac{\Psi'(\lambda)}{2\lambda\Psi(\lambda)}}\, e^{-t\Psi(\lambda)}(\Psi(\lambda))^n$ is integrable at ∞:

$$(-1)^n \frac{d^n}{dt^n} \mathbf{P}(\tau_x > t) = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{\Psi'(\lambda)}{2\lambda\Psi(\lambda)}}\, e^{-t\Psi(\lambda)}(\Psi(\lambda))^n F_\lambda(x) d\lambda$$

- For large t, the integral over $[0, C/x]$ dominates
- We have good estimates for $F_\lambda(x)$ when $\lambda x \in [0, C]$

Properties of FPT (2)

Corollary

τ_x has **ultimately completely monotone** distribution:

$(t > C(n, x, \Psi))$

$$(-1)^n \frac{d^n}{dt^n} \mathbf{P}(\tau_x > t) > 0 \qquad \text{for } t \text{ large enough}$$

Asymptotics

- $$(-1)^n \frac{d^n}{dt^n} \mathbf{P}(\tau_x > t) \sim \frac{\Gamma(n+1/2)}{\pi} \frac{V(x)}{t^{n+1/2}} \text{ as } t \to \infty$$
- $$(-1)^n \frac{d^n}{dt^n} \mathbf{P}(\tau_x > t) \sim \frac{\Gamma(n+1/2)}{\pi\,\Gamma(1+\alpha)} \frac{1}{t^{n+1/2}\sqrt{\Psi(1/x)}}$$
  as $x \to 0^+$, if $\Psi(\xi)$ is α-RV at ∞ $\square$

- Bounds with explicit constants are also available
- For $n = 0$, some of the above has been known before
- More information on $V(x)$ in the next section

Part III
Section 2

Connections with fluctuation theory

Baxter-Donsker formula

Theorem [Baxter-Donsker, 1957]

When X_t is a symmetric Lévy process:

$$\int_0^\infty \int_0^\infty e^{-\xi x - zt} \mathbf{P}(\tau_x > t) dx dt = \frac{1}{\xi\sqrt{z}} \frac{1}{(z+\Phi)^\dagger(\xi)}$$

$$= \frac{1}{\xi\sqrt{z}} \exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2+u^2} \log(z+\Psi(u)) du\right) \qquad \square$$

- There is a variant for asymmetric processes

Glen Baxter, Monroe David Donsker, 1957
On the distribution of the supremum functional for processes with stationary independent increments
Trans. Amer. Math. Soc. 85

Inversion of the Laplace transform (1)

- If $\Psi(\xi) = \Phi(\xi^2)$ for a CBF $\Phi(\xi)$ (Assumption (⚓)), then our formula for $\mathbf{P}(\tau_x > t)$ inverts the double Laplace transform in Baxter-Donsker formula
- In general, partial inverse in space is known:

$$\int_0^\infty e^{-zt}\mathbf{P}(\tau_x > t)dt = \frac{V^z(x)}{\sqrt{z}}$$

But $V^z(x)$ is not explicit:

$$V^z(x) = \mathbf{E}\left(\int_0^\infty e^{-zt}\mathbf{1}_{M_t<x}dL_t\right)$$

where L_t is the local time of $M_t - X_t$ at 0

Inversion of the Laplace transform (2)

Theorem [K-Małecki-Ryznar, 2011]

If X_t is a symmetric Lévy process and $\Psi(\xi)$ is increasing on $(0, \infty)$, then:

$$\int_0^\infty e^{-\xi x} \mathbf{P}(\tau_x > t) dx = \frac{1}{\pi} \int_0^\infty \frac{\xi}{\lambda^2 + \xi^2} \sqrt{\frac{\Psi'(\lambda)}{\Psi(\lambda)}} e^{-t\Psi(\lambda)}$$
$$\times \exp\left(\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + u^2} \log \frac{\Psi'(\lambda)(\lambda^2 - u^2)}{2\lambda(\Psi(\lambda) - \Psi(u))} du \right) d\lambda \quad \square$$

- Proof: analytic continuation, contour integration and smart substitution, rather standard
- It remained undiscovered for more than 50 years!
- This theorem is used in the 'Proof no. 2' of the formula for $\mathbf{P}(\tau_x > t)$

Increasing harmonic function (1)

- $V^z(x) = \mathbf{E}\left(\int_0^\infty e^{-zt}\mathbf{1}_{t<\tau_x} dL_t\right)$
- Let $V(x) = V^0(x) = \mathbf{E}L(\tau_x)$
- As usual, $V(x) = 0$ for $x \le 0$
- Then $V(x)$ is **harmonic** in $(0, \infty)$:

$$\mathcal{A}V(x) = 0 \qquad \text{for } x > 0$$

(under some regularity assumptions)

- It is the unique increasing harmonic function
- It already appeared twice in the slides

Increasing harmonic function (2)

- Suppose that Assumption (⚓) is satisfied
- $V(x) = \lim\limits_{\lambda \to 0^+} \dfrac{F_\lambda(x)}{\lambda \sqrt{\Psi(\lambda)}}$
- $V(x) = \lim\limits_{t \to \infty} \left(\sqrt{\pi}\, t\, \mathbf{P}(\tau_x > t)\right)$
- $\mathcal{L}V(\xi) = \dfrac{1}{\xi\, \Phi^\dagger(\xi)}$ *(this holds in greater generality)*

Bounds [K-Małecki-Ryznar, 2011], [Kim-Song-Vondraček, 2010]

If X_t is a symmetric Lévy process, and $\Psi(\xi)$, $\xi^2/\Psi(\xi)$ are increasing, then: *(more general than Assumption (⚓))*

$$\frac{2}{5} \frac{1}{\sqrt{\Psi(1/x)}} \le V(x) \le 5 \frac{1}{\sqrt{\Psi(1/x)}} \qquad \square$$

Bounds for FPT

Theorem [K-Małecki-Ryznar, 2011]

If X_t is a symmetric Lévy process, and $\Psi(\xi)$, $\xi^2/\Psi(\xi)$ are increasing, then:

$$\frac{1}{100} \min\left(1, \frac{1}{200\sqrt{t\Psi(1/x)}}\right) \le \mathbf{P}(\tau_x > t) \le \min\left(1, \frac{10}{\sqrt{t\Psi(1/x)}}\right)$$

□

- Theorem applies for all subordinate BM!
  That is, $\Psi(\xi) = \Phi(\xi^2)$ satisfies the assumptions for any Laplace exponent $\Phi(\xi)$ (not just for CBFs)
- There is a (less explicit) version for asymmetric processes
- Note: $\mathbf{P}(\tau_x \le t) = 1 - \mathbf{P}(\tau_x > t)$ is much easier

Increasing harmonic function (3)

Asymptotics [K-Małecki-Ryznar, 2011], [Kim-Song-Vondraček, 2010]

If X_t is a symmetric Lévy process, and
$\Psi(\xi)$, $\xi^2/\Psi(\xi)$ are increasing, then:

- $V(x) \sim \dfrac{1}{\Gamma(1+\alpha)\sqrt{V(1/x)}}$ as $x \to 0$
  if $\Psi(\xi)$ is α-RV at ∞
- $V(x) \sim \dfrac{1}{\Gamma(1+\alpha)\sqrt{V(1/x)}}$ as $x \to \infty$
  if $\Psi(\xi)$ is α-RV at 0^+ $\square$

- Some special cases have been known before
- Under Assumption (⚓):
  - $V(x)$ is a Bernstein function
  - explicit formula for $V(x)$ can be given
- One can obtain similar results for $V^z(x)$
- $V(x)$ is much simpler than $F_\lambda(x)$ and $\mathbf{P}(\tau_x < t)$

Part III

Section 3

Eigenvalues for intervals

Interval: idea

- Let $D = (a, b)$
- As for the BM, there are f_n and μ_n such that

$$P_t^D f_n = e^{-\mu_n t} f_n, \qquad \mathcal{A}_D f_n = \mu_n f_n$$

- f_n form a complete orthogonal set in $L^2(D)$

Guess

We should have:

$$f_n(x) \approx c_1 F_{\lambda_n}(x - a) \qquad \text{for } x \approx a$$

$$f_n(x) \approx c_2 F_{\lambda_n}(b - x) \qquad \text{for } x \approx b$$

for λ_n such that:

$$\Psi(\lambda_n) \approx \mu_n$$

Interval: sketch of the proof

(*see the next slide*)

- Define 'approximate eigenfunction' $\tilde{f}_n$ so that:

$$\tilde{f}_n(x) = F_{\lambda_n}(x-a) \qquad \text{for } x \in (a, \tfrac{2}{3}a + \tfrac{1}{3}b)$$

$$f_n(x) = (-1)^{n-1} F_{\lambda_n}(b-x) \qquad \text{for } x \in (\tfrac{1}{3}a + \tfrac{2}{3}b, b)$$

and $\tilde{f}_n$ changes 'smoothly' on $(\tfrac{2}{3}a + \tfrac{1}{3}b, \tfrac{1}{3}a + \tfrac{2}{3}b)$

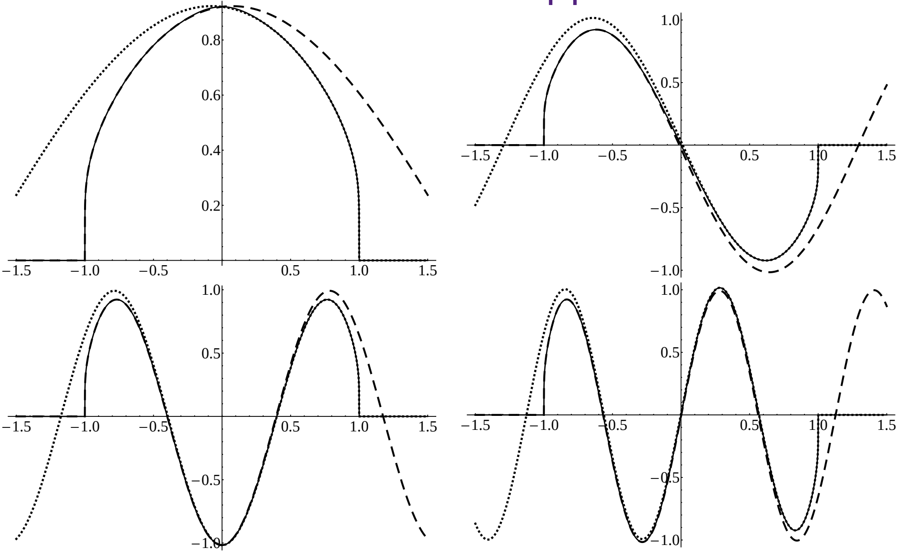
- This is possible only when:

$$\lambda_n = \frac{n\pi}{2} - \theta_{\lambda_n}$$

(then sine parts of $F_{\lambda_n}(x-a)$ and $F_{\lambda_n}(b-x)$ coincide)

- Show that $\mathcal{A}_D \tilde{f}_n \approx \Psi(\lambda_n) \tilde{f}_n$
- Deduce that $\tilde{\mu}_n \approx \Psi(\lambda_n)$

Interval: construction of the approximations



$\tilde{f}_n(x)$ (solid line), $F_{\lambda_n}(x-a)$ (dashed lined) and $\pm F_{\lambda_n}(b-x)$ (dotted line) for $\Psi(\xi) = |\xi|^{1/10}$, $D = (a, b) = (-1, 1)$, $n = 1, 2, 3, 4$.

Interval: results

Theorem [Kulczycki-K-Małecki-Stós, 2010]

For the symmetric 1-stable process, $\Psi(\xi) = |\xi|$:

$$\left| (b-a)\mu_n - \left(n\pi - \frac{\pi}{4} \right) \right| < \frac{2}{n}$$

Theorem [K, 2010]

For the symmetric α-stable process, $\Psi(\xi) = |\xi|^\alpha$:

$$(b-a)^\alpha \mu_n = \left(n\pi - \frac{(2-\alpha)\pi}{4} \right)^\alpha + O\left(\frac{1}{n}\right)$$

Theorem [Kaleta-K-Małecki]

For the relativistic process, $\Psi(\xi) = \sqrt{\xi^2 + m^2} - m$:

$$(b-a)\mu_n = \left(n\pi - \frac{\pi}{4} \right) + O\left(\frac{1}{n}\right)$$

Part III

Section 4

Higher-dimensional domains

Multidimensional domains: introduction

- Let X_t be the isotropic α-stable process in $\mathbf{R}^d$
- Let $D \subseteq \mathbf{R}^d$ be a bounded domain
- There are f_n and μ_n such that
$$P_t^D f_n = e^{-\mu_n t} f_n, \qquad \mathcal{A}_D f_n = \mu_n f_n$$
- f_n form a complete orthogonal set in $L^2(D)$
- $N(\lambda) = \#\{n : \mu_n \leq \lambda\}$ is the **partition function**

Theorem (Robert M. Blumenthal, Ronald K. Getoor, 1959)

For $C_1 = \dfrac{1}{2^d \pi^{d/2} \Gamma(d/2+1)}$:
$$\frac{N(\lambda)}{\lambda^{d/\alpha}} = C_1 |D| + o(1) \qquad \text{as } \lambda \to \infty$$
$\square$

Multidimensional domains: second term

Theorem (Rodrigo Bañuelos, Tadeusz Kulczycki, 2008)

(Abel means) As $t \to 0^+$:

$$\frac{t\,\mathcal{L}N(t)}{\Gamma(\frac{d}{\alpha}+1)t^{d/\alpha}} = C_1|D| - C_2^{(1)}|\partial D|t^{1/\alpha} + o(t^{1/\alpha}) \qquad \square$$

- $C_2^{(1)}$ given only implicitly

Theorem (Rupert L. Frank, Leander Geisinger, 2011)

(Cesaro means) As $\lambda \to \infty$:

$$\frac{\int_0^\lambda N(u)du}{(\frac{d}{\alpha}+1)\lambda^{(d+1)/\alpha}} = C_1|D| - C_2^{(2)}|\partial D|\lambda^{-1/\alpha} + o(t^{1/\alpha}) \qquad \square$$

- $C_2^{(2)}$ given explicitly in terms of the eigenfunctions $F_\lambda(x)$ for $\Psi(\xi) = (\xi^2+1)^{\alpha/2} - 1$!

Multidimensional domains: second term (2)

Conjecture

As $\lambda \to \infty$:

$$\frac{N(\lambda)}{\lambda^{d/\alpha}} = C_1 |D| - C_2^{(3)} |\partial D| t^{1/\alpha} + o(t^{1/\alpha})$$

Or even:

$$\frac{\mu_n}{\lambda^{d/\alpha}} = C_1 |D| - C_2^{(4)} |\partial D| t^{1/\alpha} + o(t^{1/\alpha})$$

- This conjecture seems to be extremely difficult
- Constants $C_2^{(n)}$ ($n = 1, 2, 3, 4$) are related to each other through simple formulae

Problems:

(1) Using the strong Markov property, prove that $(X(\tau_x + t) - X(\tau_x))$ is independent from the σ-algebra $\mathcal{F}_{\tau_x}$ and has the same law as the process X_t.

(2) Prove the reflection principle: if X_t is a symmetric Lévy process and $\mathbf{P}(X_t = 0) = 0$ for all $t > 0$, then:

$$\mathbf{P}(M_t \geq x) = \mathbf{P}(\tau_x \leq t) = 2\mathbf{P}(\tau_x \leq t, X_t \geq X_{\tau_x})$$

Prove similar *in*equalities when $\mathbf{P}(X_t = 0) > 0$ for $t > 0$.

(3) Show that for the Brownian motion:

$$\mathbf{P}(M_t \geq x) = 2\mathbf{P}(X_t \geq x)$$

(4) Show the Lévy inequality: for symmetric Lévy processes X_t:

$$\mathbf{P}(X_t \geq x) \leq \mathbf{P}(M_t \geq x) \leq 2\mathbf{P}(X_t \geq x)$$

(5) Prove that if X_t is a symmetric Lévy process and $e^{-t\Psi(\xi)}$ is integrable in $\xi \in \mathbf{R}$, then:

$$\mathbf{P}(X_t \geq x) = \frac{1}{\pi}\int_0^\infty \frac{\sin(\xi)}{\xi}(1 - e^{-t\Psi(\xi/x)})d\xi$$

Part IV

Some technical details

- Wiener-Hopf method
- Heuristic derivation of the formula for eigenfunctions

This part will be available soon

References: properties of $\Phi^\dagger(\xi)$

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References: eigenvalues in intervals

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  Spectral properties of the Cauchy process on half-line and interval
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