Liouville's theorems for Lévy operators

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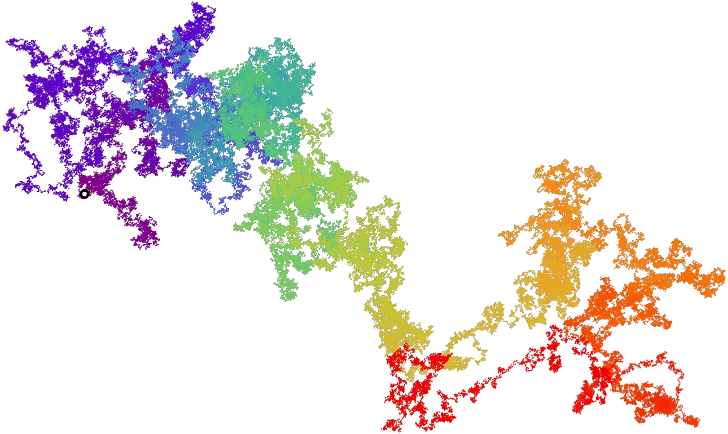
Strong Liouville's theorem

If f is a polynomially bounded harmonic function on $\mathbb{R}^d$, then f is a polynomial.

Probabilistic interpretation

Let X_t be the standard Brownian motion in $\mathbb{R}^d$, started at an arbitrary point.



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For a sufficiently regular function f, the following are equivalent:

- $\Delta f = 0$,
- $f(X_t)$ is a local martingale.

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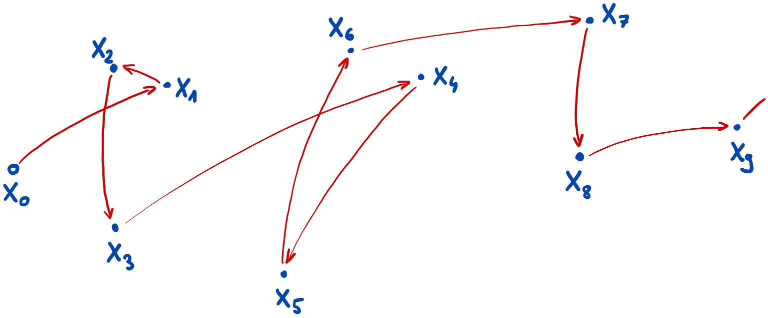
Random walks

Consider a random walk X_n in $\mathbb{R}^d$ with increments drawn from distribution ν.



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Consider a random walk X_n in $\mathbb{R}^d$ with increments drawn from distribution ν.

For a continuous function f, the following are equivalent:

- $f(x) = \int_{\mathbb{R}^d} f(x+y)\nu(dy)$,
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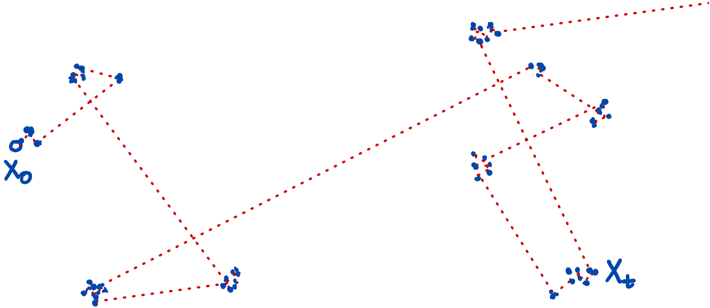
Lévy operators and Lévy processes

Consider a Lévy process X_t in $\mathbb{R}^d$ generated by a Lévy operator $\mathcal{L}$:

$$\mathcal{L}f(x) = a \cdot \nabla^2 f(x) + b \cdot \nabla f(x) + \int_{\mathbb{R}^d \setminus \{0\}} \big(f(x+z) - f(x) - z \cdot \nabla f(x) \mathbb{1}_B(z)\big) \nu(dz).$$



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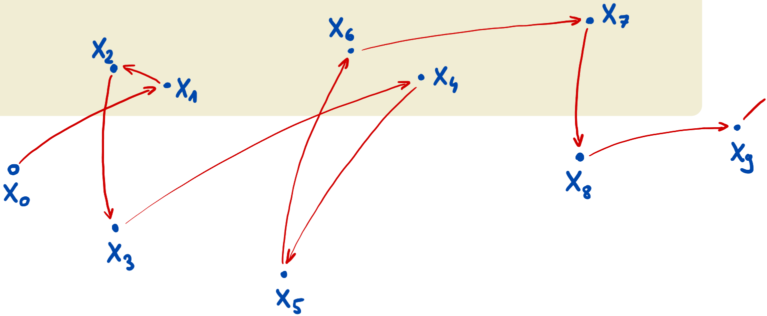

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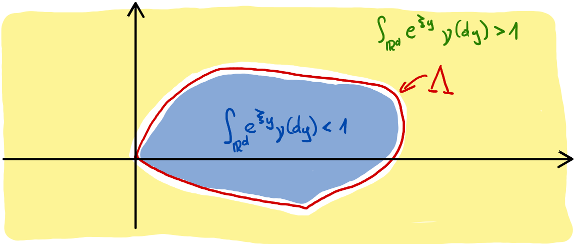

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Liouville's theorem for random walks (Deny)

A positive function f is X_n-harmonic if and only if:

$$f(x) = \int_{\Lambda} e^{\xi x} m(d\xi)$$

for a positive measure m.

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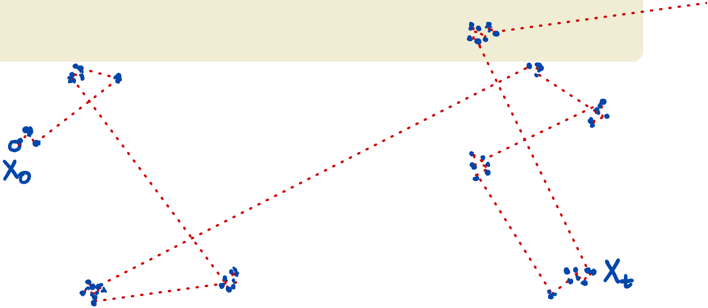

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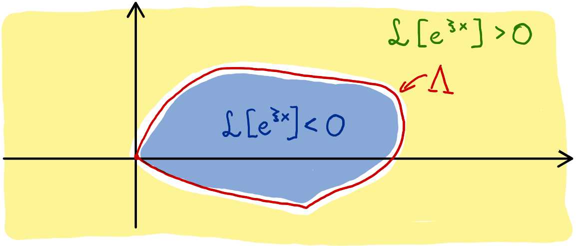

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Liouville's theorem for Lévy operators I (Berger–Schilling, TG–MK)

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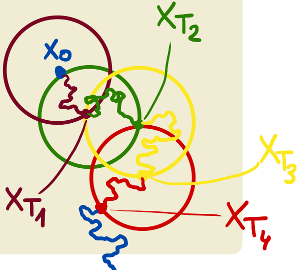

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- All that remains is a number of technical problems.

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- If F is a tempered distribution:

$$\langle \mathcal{F}F, \varphi \rangle = \langle F, \mathcal{F}\varphi \rangle \qquad \text{for every } \varphi \in \mathcal{S}.$$

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Convolution of distributions

- Tempered distributions F, G are convolvable if:

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- Bounded distributions are convolvable with integrable distributions.

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Exchange formula

If F and G are convolvable, then:

$$\mathcal{F}(F * G) = \mathcal{F}F \cdot \mathcal{F}G.$$

Fourier approach

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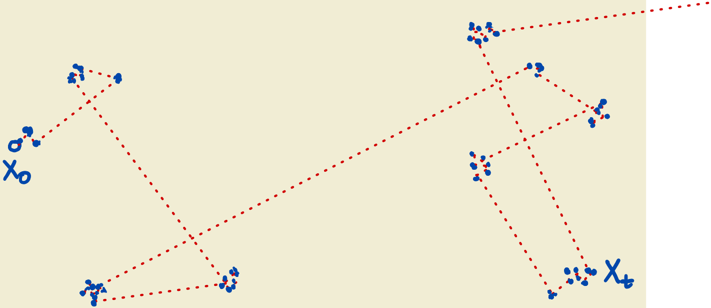

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- Thus, if f is a tempered $\mathcal{L}$-harmonic function, then:

$$\begin{aligned} \mathcal{L}f = 0 \quad & \Longrightarrow \quad L * f = 0 \\ & \Longrightarrow \quad \Psi \cdot \mathcal{F}f = 0 \\ & \Longrightarrow \quad \mathcal{F}f = 0 \quad \text{on } \mathbb{R}^d \setminus \{0\} \\ & \Longrightarrow \quad f \text{ is a polynomial.} \end{aligned}$$

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Liouville's theorem for Lévy operators II
(Chen–D'Ambrosio–Li, Fall, Berger–Schilling)

Suppose that the Fourier symbol Ψ of $\mathcal{L}$ is smooth on $\mathbb{R}^d \setminus \{0\}$.
Then every tempered $\mathcal{L}$-harmonic function f is a polynomial.
(Chen–D'Ambrosio–Li, Fall: fractional Laplacian/isotropic stable processes)

Bounded harmonic functions

$$\Psi \neq 0 \ \text{ and } \ \Psi \cdot \mathcal{F}f = 0 \quad \overset{(?)}{\Longrightarrow} \quad \mathcal{F}f = 0$$

- Trickier: Yes, if L is an integrable distribution and f is a bounded distribution.
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Liouville's theorem for Lévy operators III
(Alibaud–del Teso–Endal–Jakobsen, Berger–Schilling)

Every bounded $\mathcal{L}$-harmonic function f is constant.

Wishful thinking

$$\Psi \neq 0 \ \text{ and } \ \Psi \cdot \mathcal{F}f = 0 \quad \overset{(?)}{\Longrightarrow} \quad \mathcal{F}f = 0$$

- Natural conjecture: Yes, whenever L and f are convolvable.

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Liouville's non-theorem for Lévy operators IV (TG–MK)

For an appropriate 1-D Lévy operator $\mathcal{L}$ with positive second-order term, there is a non-polynomial, polynomially bounded $\mathcal{L}$-harmonic function.

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Corollary (TG–MK)

$$F > 0 \text{ continuous}, \quad G \text{ tempered}, \quad F \cdot G = 0 \quad \not\Longrightarrow \quad G = 0.$$

Wiener-type algebra

Definition

A Wiener-type algebra is an algebra W of continuous functions on $\mathbb{R}^d$ such that:

- every $\Psi \in W$ is a tempered distribution;
- $\varphi\Psi \in W$ whenever $\varphi \in \mathcal{S}$ and $\Psi \in W$;
- if $K \subseteq \mathbb{R}^d$ is compact, $\Psi \in W$, and $\Psi \neq 0$ on K, then for some $\Phi \in W$:

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Definition

A tempered distribution F acts on W if:

- $F \cdot (\Phi \cdot \Psi) = (F \cdot \Phi) \cdot \Psi$ whenever $\Phi, \Psi \in W$.

General result

Liouville's theorem factory for Lévy operators V (TG–MK)

Assume that the Fourier symbol Ψ of $\mathcal{L}$ belongs to W locally on $\mathbb{R}^d \setminus \{0\}$.
Then every tempered $\mathcal{L}$-harmonic function f such that $\mathcal{F}f$ acts on W is a polynomial.

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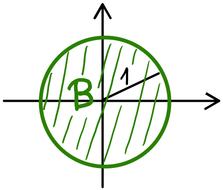
Different choices of W lead to different variants of Liouville's theorem.
We have already seen two examples: smooth symbols and bounded functions.

Applications (1/2)

Recall that ν is the non-local kernel of $\mathcal{L}$. Let B be the unit ball in $\mathbb{R}^d$.



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Liouville's theorem for Lévy operators VI
(Ros-Oton–Serra, Kühn, Berger–Schilling–Shargorodsky, TG–MK)

Assume that $|x|^\alpha \nu(dx)$ is integrable on $\mathbb{R}^d \setminus B$. If f is an $\mathcal{L}$-harmonic function such that $(1 + |x|^\alpha)^{-1} f(x)$ is bounded, then f is a polynomial.

(Ros-Oton–Serra: fractional derivatives)

Applications (1/2)

Recall that ν is the non-local kernel of $\mathcal{L}$. Let B be the unit ball in $\mathbb{R}^d$.
Let M be a positive, polynomially bounded submultiplicative function.

Liouville's theorem for Lévy operators VI
(Ros-Oton–Serra, Kühn, Berger–Schilling–Shargorodsky, TG–MK)

Assume that $M(x)\nu(dx)$ is integrable on $\mathbb{R}^d \setminus B$. If f is an $\mathcal{L}$-harmonic function such that $(M(x))^{-1} f(x)$ is bounded, then f is a polynomial.

(Ros-Oton–Serra: fractional derivatives)

(Kühn: power functions)

Applications (2/2)

Recall that ν is the non-local kernel of $\mathcal{L}$. Let B be the unit ball in $\mathbb{R}^d$.

Liouville's theorem for Lévy operators VII (Fall–Weth, TG–MK)

Assume that $|x|^{d+\alpha}\nu(dx)$ is bounded on $\mathbb{R}^d \setminus B$. If f is an $\mathcal{L}$-harmonic function such that $(1+|x|)^{-d-\alpha}f(x)$ is integrable, then f is a polynomial.

Applications (2/2)

Recall that ν is the non-local kernel of $\mathcal{L}$. Let B be the unit ball in $\mathbb{R}^d$.
Let V be a positive, integrable radial function with doubling property.

Liouville's theorem for Lévy operators VII (Fall–Weth, TG–MK)

Assume that $(V(x))^{-1}\nu(dx)$ is bounded on $\mathbb{R}^d \setminus B$. If f is an $\mathcal{L}$-harmonic function such that $V(x)f(x)$ is integrable, then f is a polynomial.

(Fall-Weth: power functions)

Counter-example: operator

Liouville's non-theorem for Lévy operators IV (TG–MK)

For an appropriate 1-D Lévy operator $\mathcal{L}$ with positive second-order term, there is a non-polynomial, polynomially bounded $\mathcal{L}$-harmonic function.

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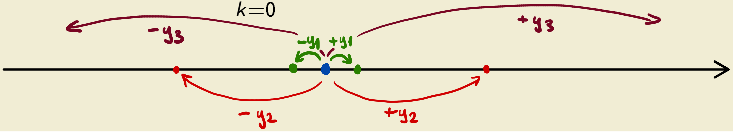
- Counter-example involves the operator:

$$\mathcal{L}f(x) = f''(x) + \sum_{k=0}^{\infty} 2^{-k}\big(f(x+y_k) + f(x-y_k) - 2f(x)\big).$$



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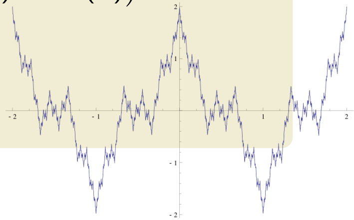
- The Fourier symbol Ψ is a Weierstrass-type function.



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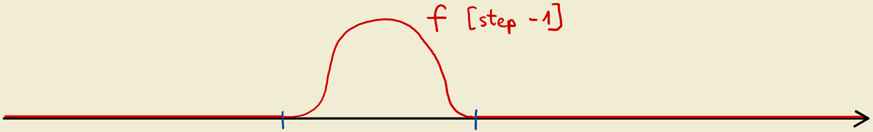

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$$I_{k+y_k} \quad \text{and} \quad I_{-k-y_k}$$

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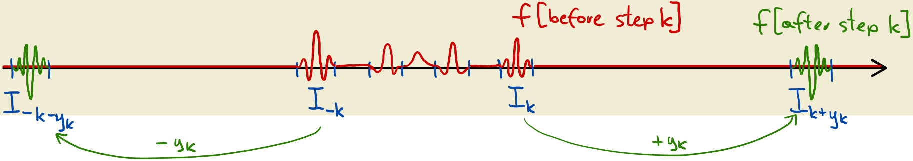

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- If y_k grows fast enough, we have $\mathcal{L} f = 0$ everywhere.
- If y_k grows really fast, $|x|^{-\varepsilon} f(x) \to 0$ as $|x| \to \infty$ for every $\varepsilon > 0$.