Lista 0. Σ notation and the binomial expansion.

1. Using sigma notation $\sum\limits_{n=...}^{...} ...$ write down the sums

   (a) $4+7+10+13+16+...+40$,

   (b) $\frac{2^2+2+1}{\ln 2}+\frac{3^2+3+1}{\ln 3}+\frac{4^2+4+1}{\ln 4}+...+\frac{70^2+70+1}{\ln 70}$,

   (c) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+...+\frac{1}{111}$,

   (d) $1+3+3^2+3^3+3^4+...+3^{20}$,

   (e) $\frac{1}{2}-\frac{1}{2^2}+\frac{1}{2^3}-\frac{1}{2^4}+\frac{1}{2^5}-\frac{1}{2^6}+...$,

   in the way that the first value of index n is

   - $n=1$,
   - $n=0$,
   - $n=4$.

2. Using sigma notation write down as a sigle sum Σ (so in a form $\sum\limits_{n=...}^{...} ...$) the sums below. Simplify your answer as far as possible.

   (a) $\sum\limits_{k=1}^{15}(k+3)^2-3\sum\limits_{k=1}^{15}(2k+3)$,

   (b) $\sum\limits_{n=0}^{9}2^n+\sum\limits_{n=1}^{10}2^n$,

   (c) $30+x^2\sum\limits_{k=1}^{10}x^k$,

   (d) $x\sum\limits_{n=0}^{10}(-1)^n\frac{x^{2n}}{(2n)!}+\sum\limits_{n=1}^{11}(-1)^n\frac{x^{2n-1}}{(2n-1)!}$, $x\neq 0$.

3. Expand and simplify

   (a) $(2x+3)^4$,

   (b) $\left(3x-\frac{1}{3x^2}\right)^5$,

   (c) $\left(\sqrt{y}+\frac{2}{y}\right)^6$,

   (d) $(a-1)^8$.

4. In the expansion of $\left(x^4 - \frac{2}{x}\right)^{135}$ find

   (a) the coefficient of x^{15},

   (b) the coefficient of $\frac{1}{x^{10}}$,

   (c) the constant term (independent of x).

5. In the expansion of $(1+4x)^n$, $n \in \mathbf{N}$, the coefficient of x^2 is 336. Find n.

6. In the expansion of $(1+ax)^n$, $n \in \mathbf{N}$, the first three terms are 1, $24x$ and $252x^2$. Find a and n.

7. Find an approximation of 1.02^{38} by taking the sum of the first four terms in the expansion of $(1+0.02)^{38}$. Find the percentage error of this approximation.

8. Using the expansion of $(1+x)^n$ show that for all $n \in \mathbf{N}_+$

   (a) $8^n + 6$ is divisible by 7,

   (b) $4^n + 2$ is divisible by 6,

   (c) $6^n + 3 \cdot 11^n + 1$ is divisible by 5,

   (d) $4^n + 6n - 1$ is divisible by 9,

   (e) $5^n + 12n + 15$ is divisible by 16,

   (f) $3^n - 2n^2 + 7$ is divisible by 8.

9. (*) Give an example of a polynomial P such that for all $n \in \mathbf{N}_+$ the sum $14^n + W(n)$ is divisible by

   (a) 13,

   (b) 169,

   (c) 2197.

10. For $n \in \mathbf{N}_+$ define the double factorial as
    $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n-1)$,
    $(2n)!! = 2 \cdot 4 \cdot 6 \cdot 8 \cdot \ldots \cdot (2n)$.

    (a) Show that $(2n-1)!! \cdot (2n)!! = (2n)!$ and $(2n)!! = n! \cdot 2^n$.

    (b) Simplify $\frac{(2n+1)!!}{(2n-1)!!}$, $\frac{(2n)!!}{(2n+2)!!}$ and $\frac{(2n-1)!!}{(2n+3)!!}$.

11. For $x \in \mathbf{R}$ and $k \in \mathbf{N}$ define the generalized binomial coefficient as
    $\binom{x}{0} = 1$ and $\binom{x}{k} = \frac{x \cdot (x-1) \cdot (x-2) \cdot \ldots \cdot (x-(k-1))}{k!}$, $k > 0$.

    (a) Show that for $x = n \in \mathbf{N}$, $n \geq k$, this definition is an extension of the basic definition of $\binom{n}{k}$.

    (b) For a given k find all x for which $\binom{x}{k} = 0$.

(c) Calculate $\binom{-1}{k}$. Simplify your answer as far as possible.

(d) For $x > 0$ and $k > 0$ show that $\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}$.

(e) Prove that $\binom{\frac{1}{2}}{k} = (-1)^{k-1} \frac{(2k-3)!!}{(2k)!!}$, $k \geq 2$,

and derive a similar formula for $\binom{-\frac{1}{2}}{k}$.

12. The Fibonacci sequence $f_n = (1, 1, 2, 3, 5, 8, 13, ...)$ is constructed by a well-known rule - the first two elements are equal to 1 and each next is a sum of the two preceeding ones. This sequence has a general formula and this formula look complicated and it is not clear, at first glance, that the elements are natural numbers:

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Using sigma notation write down this formula as a sigle sum Σ so that the terms added are rational elements (in particular, $\sqrt{5}$ must not appear).

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