# Math-algebra. Complex numbers

1. Simplify the following expressions and give the final answer in cartesian form.

   (a) $\frac{(1+2i)^2}{2+i}$.

   (b) $(2-i)^3+(2+i)^3$.

   (c) $i^n$, $n \in \mathbf{N}$.

   (d) $3i+(3i)^2+(3i)^3+...+(3i)^{51}$.

   (e) $Re(1-5i)^4+Im\left(\frac{i}{1+i}\right)$.

2. Find the modulus of the numbers below.

   (a) $-5+12i$

   (b) $\frac{1-2i}{3+i}$

   (c) $(3+i)^{110}$

   (d) $z+1-3i$, where $z=x+yi$, $x,y \in \mathbf{R}$

3. Prove that the following formulas are true for all complex $z$ and $w$.

   (a) $Re(z+w)=Re(z)+Re(w)$, $Re(z-w)=Re(z)-Re(w)$, $Im(z+w)=Im(z)+Im(w)$ and $Im(z-w)=Im(z)-Im(w)$.

   (b) $z+\overline{z}=2Re(z)$, $z-\overline{z}=2iIm(z)$ and $z\cdot\overline{z}=|z|^2$.

   (c) If $z \in \mathbf{R}$ then $Re(z\cdot w)=z\cdot Re(w)$ and $Im(z\cdot w)=z\cdot Im(w)$.

   (d) $\overline{z+w}=\overline{z}+\overline{w}$, $\overline{z-w}=\overline{z}-\overline{w}$, $\overline{z\cdot w}=\overline{z}\cdot\overline{w}$, and if $w\neq 0$ then $\overline{\left(\frac{z}{w}\right)}=\frac{\overline{z}}{\overline{w}}$.
   Then deduce that for $n \in \mathbf{N}_+$ we get $\overline{z^n}=(\overline{z})^n$.

   (e) $-|z|\leq Re(z)\leq |z|$, $-|z|\leq Im(z)\leq |z|$ and $z\in\mathbf{R} \Leftrightarrow z=\overline{z}=\pm|z|$.

   (f) $|z\cdot w|=|z|\cdot|w|$, and if $w\neq 0$ to $\left|\frac{z}{w}\right|=\frac{|z|}{|w|}$.
   Then deduce that for $n\in\mathbf{N}_+$ we get $|z^n|=|z|^n$.

   (g) $|z+w|^2=|z|^2+|w|^2+2Re(z\cdot\overline{w})$.

   (h) $|z+w|\leq|z|+|w|$.

   (i) $|z+w|=|z|+|w| \Leftrightarrow z=0 \vee w=0 \vee \frac{z}{w}>0$.

4. Solve the following complex equations and inequalities. In case of infinitely many solutions plot them in the complex plane.

   (a) $\overline{z}+2-i=(3+2i)z$.

   (b) $Re(2iz)=4Im((1-i)\overline{z})$.

   (c) $Re(z^2)\leq 0$.

   (d) $Im(z^3)=0$.

(e) $Re\left(\frac{2}{z}\right) = 0.$

(f) $Im\left(\frac{2}{z}\right) = 1.$

(g) $Im\left(\frac{1+2i}{z+1}\right) \geq 0.$

(h) $|z+5-3i| = 2.$

(i) $|iz+3+2i| > 1.$

(j) $|z+1-i| = |z+4|.$

(k) $|i\overline{z}-2+3i| \leq |z-3|.$

(l) $|z^2-4| < 5|z+2|.$

(m) $3|z-2i| < |z^2+4| \leq 2|z+2i|.$

(n) $2|z+3| = |z+6i|.$

(o) $2|z+1| \leq |\overline{z}+10+12i|.$

5. (*)

(a) Let $z_1, z_2$ be two distinct complex numbers and let $c > 0$, $c \neq 1$. Prove that the set of all $z \in \mathbf{C}$ for which $\frac{|z-z_1|}{|z-z_2|} = c$ is a circle. Find its center and radius.

(b) What kind of curve appears for $c = 1$?

6. Represent the numbers below in polar and trigonometric form.

(a) $3+3i$

(b) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(c) $-1-i\sqrt{3}$

(d) $\sqrt{5}-i\sqrt{5}$

(e) $z \in \mathbf{R}$, $z \notin 0$

(f) $z = yi$, $y \in \mathbf{R}$, $y \notin 0$

7. Find the polar and the cartesian form of the numbers below.

(a) $(1-i)^{23}.$

(b) $-(-1+i\sqrt{3})^{-14}.$

(c) $\frac{(1+i)^{39}}{1-i\sqrt{3}}.$

(d) $\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{33} \cdot (\sqrt{3}-i)^7.$

8. Using the basic version of de Moivre's theorem prove its alternative version:
for $z \in \mathbf{C}$ and $n \in \mathbf{Z}$ we have $(\,|z|(\cos\alpha - i\sin\alpha)\,)^n = |z|^n(\cos n\alpha - i\sin n\alpha).$

9. Let $z = x + yi$ and $z_0 = x_0 + y_0 i$, $x, y, x_0, y_0 \in \mathbf{R}$.

(a) For $\alpha \neq \frac{\pi}{2}$, $\alpha \neq \frac{3\pi}{2}$, find an equation of the half-line $arg(z - z_0) = \alpha$. State it in a form $y = f(x)$, $x \in S$, where $S$ is and adequate set to be found.

(b) Which half-line do we have for $\alpha = \frac{\pi}{2}$, and which for $\alpha = \frac{3\pi}{2}$?

10. (Task 4 continued). Solve the following complex equations and inequalities. In case of infinitely many solutions plot them in the complex plane.

(a) $Re\left(z(1 + i\sqrt{3})^8\right) = 128.$

(b) $Im\left((\overline{z} + 1)(-\sqrt{3} + i)^9\right) < 256.$

(c) $\left| z \left( -\frac{\sqrt{3}}{2} + \frac{1}{2} i \right)^{15} + 2 \right| \leq 5.$

(d) $arg(z + 2 - 3i) = \frac{\pi}{4}.$

(e) $\frac{\pi}{3} < arg(z - 2i) \leq \frac{5\pi}{6}.$

(f) $arg(iz) = \frac{\pi}{3}.$

(g) $\frac{\pi}{4} \leq arg\left((1 + i)(z + 2)\right) \leq \frac{5\pi}{4}.$

(h) $arg\left(\frac{-1 + i}{z}\right) = \frac{\pi}{4}.$

(i) $arg(z^3) = \frac{\pi}{2}.$

(j) $arg(-2iz^4) = \pi.$

(k) $\frac{\pi}{2} \leq arg(z^4) \leq \frac{3\pi}{4}.$

(l) $\begin{cases} |z - 3i| = 1, \\ arg(z) = \frac{\pi}{2}. \end{cases}$

(m) $\begin{cases} |z - 1| = 3, \\ arg(z + i) = \frac{\pi}{4}. \end{cases}$

(n) $\begin{cases} |z + 1| = 5, \\ arg(z + 3 - 3i) = \frac{3\pi}{4}. \end{cases}$

(o) $\begin{cases} arg(z + 7) = \frac{3\pi}{2}, \\ arg(z + 5 - i) = \frac{4\pi}{3}. \end{cases}$

(p) $\begin{cases} arg(z - 2i) = \frac{\pi}{6}, \\ arg(z - 1 - 3i) = \frac{\pi}{3}. \end{cases}$

11. Using polar form of complex numbers solve the equations below. State the solutions in cartesian form, simplifying your answers as far as possible.

   (a) $z^4 = |z|$.
   (b) $(\sqrt{3} + i)(\overline{z})^3 = z$.
   (c) $|z|^5 = iz^5$.

12. (*) Assume that $z, w \in \mathbf{C} \setminus \{0\}$, $z + w \neq 0$ and $|z| = |w|$.

   (a) Show that it is always possible to find such an argument $\alpha$ of $z$ and argument $\beta$ of $w$ that $|\alpha - \beta| < \pi$ (they need not be principal arguments).
   (b) Prove by any method that for such $\alpha$ and $\beta$ the angle $\varphi = \dfrac{\alpha + \beta}{2}$ is an argument of $z + w$.

13. Using the previous task represent the numbers below in cartesian and polar form.

   (a) $-\sqrt{2} + 1 + i$.
   (b) $1 + i(\sqrt{3} + 2)$.
   (c) $1 + i(\sqrt{3} - 2)$.
   (d) $1 + i(2 - \sqrt{3})$.
   (e) $\sqrt{2} + 1 + i(\sqrt{2} + \sqrt{3})$.
   (f) $\cos\frac{\pi}{9} + \cos\frac{\pi}{7} + i\left(\sin\frac{\pi}{9} + i\sin\frac{\pi}{7}\right)$.
   (g) $\cos\frac{\pi}{9} - \cos\frac{\pi}{7} + i\left(\sin\frac{\pi}{9} - i\sin\frac{\pi}{7}\right)$.

14. Find the following roots of complex numbers. State the answers in cartesian form.

   (a) $\sqrt{-3 + 4i}$
   (b) $\sqrt{7 + i}$
   (c) $\sqrt[3]{8i}$
   (d) $\sqrt[3]{-27}$
   (e) $\sqrt[4]{-1}$
   (f) $\sqrt[4]{-1 + i\sqrt{3}}$
   (g) $\sqrt[5]{1}$
   (h) $\sqrt[4]{i}$
   (i) $\sqrt[3]{1 + i}$
   (j) $\sqrt[4]{(1 + 2i)^4}$
   (k) $\sqrt[3]{(1 + 2i)^6}$
   (l) $\sqrt[n]{z^n}$, $z \in \mathbf{C}$, $n \in \mathbf{N}_+$

15. (*) Verify that

   (a) if $z < 0$ then $\sqrt{z} = \pm i\sqrt{|z|} = \pm i\sqrt{-z}$,

(b) if $z \notin \mathbf{R}$ then $\sqrt{z} = \pm\sqrt{|z|} \cdot \dfrac{z+|z|}{|z+|z||}$.

16. Let $z \in \mathbf{C}$, $z \neq 0$, and $m, n \in \mathbf{N}$, $m, n > 1$. Show that the set $\sqrt[n]{z}$ is a subset of the set $\sqrt[mn]{z^m}$ but these sets are not the same.

17. Using complex roots solve the equations below.

    (a) $az^2 + bz + c = 0$, $a, b, c \in \mathbf{R}$, $a \neq 0$, $\Delta = b^2 - 4ac < 0$
    (b) $z^2 + (1+i)z - 2 - i = 0$
    (c) $iz^2 + (2+i)z + 5 - i = 0$
    (d) $z^3 = (2z+i)^3$
    (e) $z^4 = -4(z+1)^4$

18. (*) Let $z$, $z_0 \in \mathbf{C}$, $\alpha \in \mathbf{R}$. Define two complex functions $f$, $g$ as $w = f(z) = ze^{i\alpha}$ and $w = g(z) = z_0 + (z - z_0)e^{i\alpha}$. Prove their geometrical properties:

    (a) $f$ is a rotation around the center, by angle $\alpha$,
    (b) $g$ is a rotation around the point represented by $z_0$, by angle $\alpha$.

    As usually, for $\alpha > 0$ the rotations are anticlockwise, and clockwise for $\alpha < 0$.

    Using this method find the image of $P = (1, 3)$ when $P$ is rotated

    (a) around the origin by $\alpha = 30°$ anticlockwise,
    (b) around $Q = (2, -1)$ by $\alpha = 135°$ clockwise.

19. (*) Recall that $\arctan A$ is the angle $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which $\tan\alpha = A$.

    Using complex numbers show that

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

    This result, combined with a Maclaurin's series of $\arctan x$, enables us to find, effectively, an approximation of $\pi$, with high accuracy.

20. Using complex numbers derive the formula of $\sin(5x)$ in terms of $\sin x$ only.

21. Using complex numbers find a formula of

    (a) $\cos x + \cos(2x) + \ldots + \cos(nx)$,
    (b) $2\sin x + 4\sin(2x) + \ldots + 2^n \sin(nx)$.

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