Math-algebra. Polynomials and rational functions

1. Divide P by Q if

   (a) $P(x) = 3x^6 + x^3 + x + 1$, $Q(x) = x^3 - x + 1$,
   (b) $P(x) = x^2 + 5x + 1$, $Q(x) = -2x^2 + 5$,
   (c) $P(x) = 2x^3 + x^2 + x - 7$, $Q(x) = 3x + 5$,
   (d) $P(x) = x^3 - 3x + 2$, $Q(x) = x + 2$.

2. (a) $P(x) = 4x^{211} + cx^3 - x^2 + 1$ is divisible by $Q(x) = x + 1$. Find c.
   (b) $P(x) = ax^3 + bx^2 + x - 2$ has a root $x = 1$ and leaves a reminder -5 when divided by $Q(x) = x - 2$. Find a and b.

3. The following polynomials have integer repeated roots. Find their multiplicity.

   (a) $P(x) = x^5 - 4x^4 + 3x^3 + 5x^2 - 8x + 4$
   (b) $P(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$
   (c) $P(x) = x^6 - 4x^5 + 7x^4 - 8x^3 + 7x^2 - 4x + 1$
   (d) $P(x) = x^7 + 3x^6 + 3x^5 + 2x^4 + 5x^3 + 9x^2 + 7x + 2$

4. Factorize the polynomials below over the

   - real number field,
   - complex number field.

   Use any reasonable methods: finding integer/rational/complex roots, grouping suitable terms, substituting a new variable etc.

   (a) $-x^3 + 19x - 30$
   (b) $2x^3 + 3x^2 + 2x + 1$
   (c) $-\frac{1}{2}x^3 + x - \frac{1}{2}$
   (d) $5x^3 - 24x^2 + 36x - 16$
   (e) $\frac{1}{9}x^3 - x^2 + 3x - 3$
   (f) $x^4 - 3x^3 - 4x^2 + 18x - 12$
   (g) $x^4 + x^3 - x - 1$
   (h) $-9x^4 - 3x^3 + 23x^2 - 13x + 2$
   (i) $8x^4 + 20x^3 - 42x^2 + 23x - 4$
   (j) $16x^4 + 32x^3 + 24x^2 + 8x + 1$
   (k) $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$
   (l) $x^5 - x^4 - 2x^3 + 5x^2 - 5x + 2$
   (m) $x^6 + 5x^5 + 7x^4 - 2x^3 - 13x^2 - 11x - 3$
   (n) ($\star$) $x^5 + 2$
   (o) ($\star$) $x^7 - 1$

5. Factorize the following quartic polynomials over the real number field.

(a) $x^4 - 7x^2 + 5$
(b) $-x^4 - x^2 + 2$
(c) $2x^4 + 3x^2 + 1$
(d) $x^4 - 6x^2 + 9$
(e) $\frac{1}{2}x^4 + x^2 + \frac{1}{2}$
(f) $x^4 + 9$
(g) $x^4 - x^2 + 25$
(h) $3x^4 + 5x^2 + 9$

6. (*)

(a) Prove that if z is a root of a polynomial with real coefficients then $\overline{z}$ is also a root of this polynomial.
(b) Prove that the multiplicities of z and $\overline{z}$ are the same.

7. $x = -2 + i$ is a root of $P(x) = -x^4 - 3x^3 - 3x^2 - 3x - 10$. Factorize P over the

- real number field,
- complex number field.

8. $x = i\sqrt{3}$ is a double root of $P(x) = 2x^6 + x^5 + 13x^4 + 6x^3 + 24x^2 + 9x + 9$. Factorize P over the

- real number field,
- complex number field.

9. (a) Using the formula $\sin(5x) = 16\sin^5 x - 20\sin^3 x + 5\sin x$ (see List 1) find the exact values of $\sin\frac{\pi}{5}$, $\sin\frac{2\pi}{5}$, $\sin\frac{3\pi}{5}$ and $\sin\frac{4\pi}{5}$.
(b) Deduce that $\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$ and state in a similar form $\cos\frac{2\pi}{5}$, $\cos\frac{3\pi}{5}$ oraz $\cos\frac{4\pi}{5}$.

10. Using the factor form of real polynomials prove that every real polynomial has at least one real root. Deduce then that the range of such polynomials is $\mathbf{R}$.

11. Let $P = P(x)$ be a complex polynomial. Let $Q = Q(x) = x - x_0$, $x_0 \in \mathbf{C}$, be its factor of multiplicity $k \in \mathbf{N}_+$. Let $W(x) = P'(x)$ be the derivative of P (the definition of derivative, derivatives of basic functions and rules of differentiation are he same as in real case). Prove the followin theorem.

(a) $k = 1 \iff Q$ is not a factor of W.
(b) $k > 1 \iff Q$ is a factor of W, with multiplicity $k - 1$.

12. (*) Using the previous task prove the following theorem.

   Let $P = P(x)$ be a complex polynomial and let $x_0 \in \mathbf{C}$ be its root. Then

   x_0 is of multiplicity $k \in \mathbf{N}_+ \iff P(x_0) = P'(x_0) = P''(x_0) = ... = P^{(k-1)}(x_0) = 0$ oraz $P^{(k)}(x_0) \neq 0$

   (the first derivative that is not equal to 0 at x_0 is the kth one).

   This theorem enables us to find multiplicity of x_0 without repeated division. Furthermore, it allows to fint this multiplicity in case when x_0 cannot be calculated explicitely.

13. Find the remainder when $P(x) = 2x^{151} + 4x - 1$ is divided by

   (a) $Q(x) = x^2 + x - 2$,
   (b) $Q(x) = x^3 - 4x$,
   (c) $Q(x) = x^2 + 1$,
   (d) $Q(x) = x^2 + 2x + 2$,
   (e) $Q(x) = x^2 + x\sqrt{3} + 1$,
   (f) $Q(x) = x^2 - 2x + 4$,
   (g) (*) $Q(x) = x^3 - 3x + 2$,
   (h) (*) $Q(x) = x^4 - 3x^3 + 3x^2 - x$.

14. Represent the functions below as a sum of partial fractions, without calculating their coefficients.

   (a) $f(x) = \dfrac{x-3}{(x-1)^3(2x-1)(x^2+x+3)(x^2+3)^2}$
   (b) $f(x) = \dfrac{x^2}{(x^2+4x+3)(x^2+4x+4)(x^2+4x+5)}$
   (c) $f(x) = \dfrac{x^3}{(x^2-5)^3(x^2+5)^3}$

15. Decompose the functions below into real partial fractions.

   (a) $f(x) = \dfrac{2x+1}{x^2-4x+3}$
   (b) $f(x) = \dfrac{x^2}{x^3+2x^2-x-2}$
   (c) $f(x) = \dfrac{5}{x^3-3x+2}$
   (d) $f(x) = \dfrac{x^2-4x-3}{x^3+x+2}$
   (e) $f(x) = \dfrac{-4x^2+26x-34}{x^4-8x^3+18x^2-27}$
   (f) $f(x) = \dfrac{2x^3+4x^2+6x+9}{x^4+5x^2+6}$
   (g) $f(x) = \dfrac{x^3+2x+1}{x^4+2x^2+1}$

16. Represent the functions below as a sum of a polynomial and real partial fractions.

 (a) $f(x) = \dfrac{5x^5 + x - 1}{x^3 + x^2}$

 (b) $f(x) = \dfrac{4x^2 + x - 1}{x^2 - 3x + 2}$

17. (*) Consider a proper rational function $f(x) = \dfrac{P(x)}{Q(x)}$ such that polynomials P and Q don't have common factors.

 (a) Prove that its partial fraction decomposition must contain the fractions with the highest possible power of the denominator.

 For example, in case of $\dfrac{P(x)}{x^3(x^2+1)^2}$ the fractions $\dfrac{A}{x^3}$ and $\dfrac{ax+b}{(x^2+1)^2}$ must occur while $\dfrac{B}{x^2}$, $\dfrac{C}{x}$ and $\dfrac{Dx+E}{x^2+1}$ need not appear.

 (b) Prove that for a partial fraction with a denominator that is a linear function to the highest possible power, a popular method of finding its numerator - multiply the equation

 $\dfrac{P(x)}{Q(x)} =$ sum of partial fractions

 by Q and input at x a suitable root of Q - always leads to a linear equation with one variable and this variable is the desider numerator.

 Using that approach derive the formula of such a partial fraction.

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