1. Improper integrals.

1. Find the exact values of the integrals below.

   (a) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}(x+3)}dx.$

   (b) $\displaystyle\int_1^{\infty} \frac{1}{(\sqrt[3]{x}+2)^2}dx.$

   (c) $\displaystyle\int_0^{\infty} \frac{2^x}{4^x+1}dx.$

   (d) $\displaystyle\int_{-\infty}^{0} \frac{1}{x^2-3x+2}dx.$

   (e) $\displaystyle\int_{-\infty}^{1} xe^{2x}dx.$

   (f) $\displaystyle\int_1^{\infty} \frac{x+1}{2x^3+x^2}dx.$

   (g) $\displaystyle\int_1^{\infty} \frac{\ln x}{(x+1)^2}dx.$

   (h) $\displaystyle\int_0^{\infty} e^{-x}\sin x dx.$

   (i) $\displaystyle\int_{-\infty}^{\infty} e^{-|x+2|}dx.$

   (j) $\displaystyle\int_{-\infty}^{\infty} \frac{x^3+1}{(x^4+4x+7)^2}dx.$

2. Derive the formulas of the integrals below.

   (a) $\displaystyle\int_a^{\infty} A^x dx$, and $\displaystyle\int_{-\infty}^{a} A^x dx$ for $a \in \mathbf{R}$ and $A > 0$.

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{ax^2+bx+c}dx$ for $a \neq 0$, $b, c \in \mathbf{R}$ and $\Delta = b^2 - 4ac < 0$.

   (c) $\displaystyle\int_a^{\infty} \frac{1}{(x-x_1)(x-x_2)}dx$ for $a, x_1, x_2 \in \mathbf{R}$ such that $x_1 < x_2 < a$.

(d) $\displaystyle\int_a^\infty \frac{1}{x\ln^p x}dx$ for $a, p > 0$.

(e) $\displaystyle\int_a^\infty \frac{\ln x}{x^p}dx$ for $a, p > 0$.

3. Formulas like 'integral of sum/difference = sum/difference of integrals" (valid for definite integrals) need not be true for improper integrals. Show that if $f(x) = \frac{1}{x}$ oraz $g(x) = \frac{1}{x-1}$ then

$$\int_2^\infty (f(x) + g(x))dx = \int_2^\infty f(x)dx + \int_2^\infty g(x)dx$$

but the formula

$$\int_2^\infty (f(x) - g(x))dx = \int_2^\infty f(x)dx - \int_2^\infty g(x)dx$$

is not true.

4. Prove that if an integral $\displaystyle\int_{-\infty}^a f(x)dx$ has a finite or infinite value then $\displaystyle\int_{-\infty}^a f(x)dx = \int_{-a}^\infty f(-x)dx$.

5. Prove that if $F' = f$ na $\mathbf{R}$ then

$$\int_{-\infty}^\infty f(x)dx = \lim_{T\to\infty} F(T) - \lim_{S\to-\infty} F(S),$$

under the assumption that the right-hand side is well-defined.

This formula enables us to simplify the calculations when evaluating integrals over $\mathbf{R}$.

6. Prove that for any f that is integrable over each interval of a form $[a, T]$

(a) if $\lim_{x\to\infty} f(x) > 0$ then $\displaystyle\int_a^\infty f(x)dx = \infty$,

(b) if $\lim_{x\to\infty} f(x) < 0$ then $\displaystyle\int_a^\infty f(x)dx = -\infty$.

7. (*) Give examples of elementary functions that are continuous on $[a, \infty)$ and $\lim_{x\to\infty} f(x)$ does not exist, while the value of integral $\displaystyle\int_a^\infty f(x)dx$

(a) is finite,

(b) is infinite,

(c) does not exist.

8. Using suitable tests test convergence of the integrals below.

(a) $\int\limits_1^\infty \frac{2x^2+x+1}{x^4+x+1}dx.$

(b) $\int\limits_1^\infty \frac{2x^3+x+1}{x^4+x+1}dx.$

(c) $\int\limits_1^\infty \frac{2^x-1}{5^x-1}dx.$

(d) $\int\limits_{10}^\infty \frac{5\cdot 3^x+2^x+1}{4^x-2\cdot 3^x-7}dx.$

(e) $\int\limits_3^\infty \frac{\sqrt[5]{x^6+x+2}}{\sqrt{2x^4-x^3+2}}dx.$

(f) $\int\limits_1^\infty \left(\sqrt{x^4+1}-x^2\right)dx.$

(g) $\int\limits_1^\infty \left(\sqrt{x^4+x+1}-x^2\right)dx.$

(h) $\int\limits_1^\infty \left(1-\frac{2}{x}\right)^x dx.$

(i) $\int\limits_1^\infty (\ln(2x+1)-\ln x)\,dx.$

(j) $\int\limits_1^\infty \frac{\ln x}{\sqrt{3x+5}}dx.$

(k) $\int\limits_\pi^\infty \frac{x^2+\cos x}{x^3+x^2\cdot\sqrt{x}}dx.$

(l) $\int\limits_\pi^\infty \frac{1+\cos x}{x+\sqrt{x}}dx.$

(m) (*) $\int\limits_\pi^\infty x^2\left(1-\cos\frac{1}{x}\right)dx.$

9. (previous problem continued) Using suitable tests test convergence of the integrals below.

(a) $\int\limits_1^\infty \sin\frac{1}{\sqrt[3]{x}}dx.$

(b) $\int\limits_1^\infty \sin^4 \frac{1}{\sqrt[3]{x}} dx$.

(c) $\int\limits_1^\infty \sqrt[3]{\sin \frac{1}{x^4}} dx$.

(d) $\int\limits_1^\infty \ln\left(1 + \frac{3}{2^x}\right) dx$.

(e) $\int\limits_1^\infty x^2 \cdot \mathrm{tg} \frac{1}{x^3} dx$.

(f) $\int\limits_1^\infty 2^x \cdot \mathrm{arctg} \frac{1}{3^x} dx$.

(g) $\int\limits_1^\infty \sin \frac{1}{\sqrt[3]{x}} \cdot \arcsin \frac{1}{\sqrt{x}} dx$.

(h) (*) $\int\limits_1^\infty \left(e^{\frac{1}{x}} - 1 - \frac{1}{x}\right) dx$.

10. Areas and volumes of unbounded regions, and their finiteness.

  (a) Evaluate, exactly, the area of the region that is bounded by the axes and the curve $f(x) = e^{-\sqrt{x}}$.

  (b) The region bounded by the X-axis and the graph of $y = \frac{\sqrt{\ln x}}{\sqrt[3]{x^2}}$ is rotated around the X-axis and generates a circular solid. Find, exactly, its volume.

  (c) Show that even though the volume in b) is finite, the region has an infinite area.

11. Show that the integrals below are absolutely convergent.

  (a) $\int\limits_1^\infty \frac{\sin(x^2)}{x\sqrt[3]{x} + 2} dx$.

  (b) $\int\limits_{2\pi}^\infty \frac{4\cos x - 3}{2^x - 1} dx$.

  (c) $\int\limits_\pi^\infty \frac{\sin(2x)}{x^2 + \cos(5x)} dx$.

  (d) $\int\limits_0^\infty \frac{3^x \cos x}{2^x + 5^x} dx$.

12. (*) Prove that for all $A \neq 0$, $B \in \mathbf{R}$ and $a > 0$, the integral $\int\limits_a^\infty \frac{\cos(Ax + B)}{x} dx$ is conditionally convergent.

13. Find Cauchy principal values of the integrals below.

(a) $$\int_{-\infty}^{\infty} \frac{x^3 + \sin(2x)}{x\sqrt[3]{x} + 2} dx.$$

(b) $$\int_{-\infty}^{\infty} \frac{x^2}{x^6 + 1} dx.$$

(c) $$\int_{-\infty}^{\infty} x \cdot |x+1| dx.$$

(d) $$\int_{-\infty}^{\infty} (-x^2 + x + 1 + |x^2 - 1|) dx.$$

(e) $$\int_{-\infty}^{\infty} \sin\left(x + \frac{\pi}{6}\right) dx.$$

14. Calculate the following type 2 improper integrals.

(a) $$\int_{0}^{64} \frac{\sqrt{x} + 2\sqrt[3]{x}}{x} dx.$$

(b) $$\int_{0}^{1} \frac{\ln x}{\sqrt[3]{x}} dx.$$

(c) $$\int_{1}^{2} \frac{2^x}{2^x - 4} dx.$$

(d) $$\int_{1}^{4} \frac{1}{2 - \sqrt{x}} dx.$$

(e) $$\int_{-1}^{1} \frac{1}{x^2} e^{\frac{1}{x}} dx.$$

15. There are many ways to convert any type 2 improper integral into some of type 1. Prove the following two possible.

(a) If f is continuous on $(a, b]$ then
$$\int_{a}^{b} f(x)dx = \int_{c}^{\infty} \frac{1}{t^2} f\left(a + \frac{1}{t}\right) dt, \text{ where } c = \frac{1}{b-a}.$$

(b) If f is continuous on $[a, b)$ then
$$\int_{a}^{b} f(x)dx = \int_{c}^{\infty} \frac{1}{t^2} f\left(b - \frac{1}{t}\right) dt, \text{ where } c = \frac{1}{b-a}.$$

Krzysztof „El Profe” Michalik