2. Series.

1. Using the form of telescoping series find, exactly, the sums below. Simplify your answers as far as possible.

   (a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2+1}-\frac{1}{(n+1)^2+1}\right).$

   (b) $\displaystyle\sum_{n=0}^{\infty}\left(\arccos\left(\frac{1}{n+1}\right)-\arccos\left(\frac{1}{n+3}\right)\right).$

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+3n}.$

   (d) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}.$

   (e) $\displaystyle\sum_{n=2}^{\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\ln(n+1)\cdot\ln n}.$

2. (*) Consider two numbers $x, y \in \mathbf{R}$ such that $xy \neq -1$.

   (a) Prove that x and y are of the same sign then

   $\operatorname{arctg}x - \operatorname{arctg}y = \operatorname{arctg}\left(\dfrac{x-y}{1+xy}\right),$

   and show such x and y for which the formula is false.

   (b) Using this formula find, exactly, the sum of $\displaystyle\sum_{n=1}^{\infty}\operatorname{arctg}\left(\frac{2}{n^2}\right),$

   simplifying your answers as far as possible.

3. Using adequate tests, test convergence of the following series.

   (a) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{2n^2+n-3}+\sqrt[3]{n^5}}{n^3+2}.$

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(2n^2+n+3)^{30}+1}{n!}.$

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+3^n}{2^n+n^2\cdot 3^n}.$

   (d) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+1}{5n+2}\right)^n.$

   (e) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5n+1}{5n+2}\right)^n.$

   (f) $\displaystyle\sum_{n=1}^{\infty}\left(\arccos\left(\frac{5n+1}{5n+2}\right)\right)^n.$

(g) $\sum_{n=1}^{\infty} \operatorname{arctg}\left(\frac{1}{\sqrt{n}}\right)$.

(h) $\sum_{n=1}^{\infty} \frac{3+\cos(n^2)}{\sqrt[5]{n^3}}$.

(i) $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{(\ln 2)^n + (\ln 3)^n}$.

(j) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{n^2+200}$.

(k) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{n^3+200}$.

(l) $\sum_{n=1}^{\infty} (-1)^n \cdot (\sqrt[3]{n+7} - \sqrt[3]{n})$.

(m) $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n)}{\sqrt{2^n - 1}}$.

4. (previous problem continued) Test convergence of the following series.

   (a) $\sum_{n=1}^{\infty} \sqrt{\sin \frac{1}{n^2}}$.

   (b) $\sum_{n=1}^{\infty} 7^n \arcsin^2\left(\frac{1}{3^n}\right)$.

   (c) $\sum_{n=2}^{\infty} \sqrt[4]{n} \cdot \operatorname{tg}\left(\frac{1}{\sqrt[3]{n}}\right) \cdot \left(\sqrt[n]{5} - 1\right)$.

   (d) (*) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt[3]{n}} - \operatorname{arctg}\left(\frac{1}{\sqrt[3]{n}}\right)\right)$.

5. Using the following estimate for natural logarithm

$$\forall p > 0\ \exists C > 0\ \forall n > 2\ \ 1 \leq \ln n \leq Cn^p$$

   test convergence of the following series.

   (a) $\sum_{n=3}^{\infty} \frac{1}{n^2 \ln n}$.

   (b) $\sum_{n=3}^{\infty} \frac{\ln^2 n}{n \cdot \sqrt[3]{n^4}}$.

   (c) $\sum_{n=3}^{\infty} \frac{\sqrt{n} - \sqrt{\ln n}}{n^2}$.

   (d) $\sum_{n=3}^{\infty} \frac{\ln^p n}{n^q}$ dla $p, q > 0$.

(e) $\sum_{n=3}^{\infty} \frac{1}{\ln^p n \cdot n^q}$ dla $p, q > 0,\ q \neq 1$.

6. (missing case of the previous problem) Show that for

$$\sum_{n=3}^{\infty} \frac{1}{n \cdot \ln^p n},\ p > 0,$$

it is not possible to use the method of the previous problem.

Test convergence of this series using another test.

7. Niech $a > 1$ oraz $p \in \mathbf{R}$. Analysing suitable series prove that $\frac{a^n}{n!}$ and $\frac{n^p}{a^n}$ tend to 0.

Conclude that also $\frac{n^p}{n!}$ tends to 0 while $\frac{n!}{a^n}$, $\frac{a^n}{n^p}$ and $\frac{n!}{n^p}$ tend to ∞.

8. (*) Using Raabe's test (individual investigation expected) show that

   (a) $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!! \cdot (2n+1)}$ is a convergent series,

   (b) szereg $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$ is a dinvergent series,

   where $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$ and $(2n)!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)$.

   The result in (a) gives convergence for the power series of $\arcsin x$ at $x = \pm 1$.

9. Show that $\sum_{n=2}^{\infty} \frac{1 + 2 \cdot (-1)^n}{n}$ is an alternating series whose general term tends to 0 but the sum of this series is infinite

   Conclude that the general term of this series is not monotonic.

10. (*) Prove the folowing theorem.

    Consider an alternating series $\sum_{n=n_0}^{\infty} (-1)^n a_n$, $a_n > 0$. If a_n is non-decreasing then the sum of the series does not exist.

    This implies that if for a given alternating series ratio test gives divergence then the sum of the series does not exist.

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