3. Power series.

1. Find all $x \in \mathbf{R}$ for which the series below are convergent (you may, first, look at problem 2).

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$.

   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$.

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt[4]{n} \cdot 2^n - 1}$.

   (e) $\displaystyle\sum_{n=0}^{\infty} (2x+1)^n (2^n + 3^n)$.

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^{2n+1}}{n^2 \cdot 4^n + n \cdot 3^n + 5}$.

   (g) $\displaystyle\sum_{n=0}^{\infty} (x+2)^n \left(\frac{n+2}{n}\right)^{\frac{n^2+5}{\sqrt{n}+5}}$.

   (h) $\displaystyle\sum_{n=0}^{\infty} (x+2)^n \left(\frac{n-2}{n}\right)^{\frac{n^2+5}{\sqrt{n}+5}}$.

   (i) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{\ln(2n+1)}$.

   (j) $\displaystyle\sum_{n=0}^{\infty} 2^n \cdot x^{n^5+n+1}$.

2. A convenient theorem that may simplify the problem of investigating convergence of power series.

   (a) Prove that if a power series at one of the ends of the interval of convergence is a convergent series with nonnegative general term then it is convergent at the second end of this interval.

   (b) Give examples which show that if this series at one of the ends of the interval of convergence is a divergent series with nonnegative general term then at the second end of the interval of convergence we know nothing about the convergence of the series.

3. Using Maclaurin series of standard elementary functions find Maclaurin series of the functions below, clearly stating their interval of convergence.

   Simplify the results as far as possible and state them

   - with the use of $\sum$ notation,
   - in an expanded form, giving the first three terms of the expansions.

(a) $f(x) = xe^{2x} + \ln(1-x)$.

(b) $f(x) = \sin x - \dfrac{1}{3}\sin(3x)$.

(c) $f(x) = \ln(1+4x^2)$.

(d) $f(x) = \ln(4-x^2)$.

(e) $f(x) = \operatorname{arctg}(5x^3)$.

(f) $f(x) = \dfrac{1}{2+x^2}$.

(g) $f(x) = \dfrac{x}{8-x^3}$.

(h) $f(x) = \dfrac{1}{(1+x^2)^2}$.

(i) $f(x) = \dfrac{1}{(2-x)^3}$.

4. Using partial fraction decomposition and suitable Maclaurin expansion find Maclaurin series of the functions below.

(a) $f(x) = \dfrac{3x-5}{x^2-3x+2}$.

(b) $f(x) = \dfrac{x^2+2x}{x^3+x^2+4x+4}$.

5. Using suitable Maclaurin series state the exact values of the sums below.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)}$.

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cdot 3^n}$.

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-\pi^2)^n}{(2n+1)!}$.

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-\pi^2)^n}{2^{4n}(2n)!}$.

6. Prove that for $x \in (-1,1)$,

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \sum_{n=2}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}x^n.$$

7. (a) Applying the property of differentiation or integration to a suitable Maclaurin series find the sum of $\sum_{n=2}^{\infty} n(n-1)x^{n-2}$, $x \in (-1,1)$.
   (b) Derive the formula in (a) directly, usin the expansion of $(1+x)^p$, for a suitable p.

8. Find Maclaurin series of indefininite integrals $F(x) = \int f(x)dx$, with f given below.

   (a) $f(x) = e^{-\frac{1}{2}x^2}$.
   (b) $f(x) = x\cos(x^3)$.
   (c) $f(x) = \ln(1-x^{12})$, $x \in (-1,1)$.
   (d) $f(x) = \sqrt[3]{1+x^7}$, $x \in (-1,1)$.

9. Find series whose sums are equal to the following definite integrals.

   (a) $f(x) = \int\limits_{1}^{2} e^{x^3}\,dx$.

   (b) $f(x) = \int\limits_{\frac{\pi}{2}}^{\pi} x^2\sin(x^2)dx$.

   (c) $f(x) = \int\limits_{0}^{\frac{1}{2}} \sqrt[4]{1+x^4}dx$.

10. Using Abel's theorem for power series prove convergence of Maclaurin series for the following functions, at given ends of their interval of convergence.

   (a) $\ln(1+x)$, at $x = 1$.
   (b) $\arctan x$, at $x = \pm 1$.
   (c) ($\star$) $\sqrt{1+x}$, at $x = \pm 1$.

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