4. Multivariable functions.

1. For each set given below identify its interior, exterior and boundary. Next, verify whether the sets are

   - open,
   - closed,
   - boundary sets,
   - connected,
   - bounded,
   - open regions or closed regions.

   (a) $A = \{(x, y) \in \mathbf{R}^2 : \ 1 < (x-1)^2 + (y+5)^2 \leq 2\}$.
   (b) $B = \{(x, y, z) \in \mathbf{R}^3 : \ 0 < x^2 + y^2 + z^2 < 4\}$.
   (c) $C = \{(x, y) \in \mathbf{R}^2 : \ |x| \geq 2, \ |y| \leq 4\}$.
   (d) $D = \left\{x \in \mathbf{R} : \ \frac{1}{x} < 5\right\}$.
   (e) $E = \left\{(x, y, z) \in \mathbf{R}^3 : \ x = \frac{1}{n}, y = \frac{2}{n}, z = \frac{3}{n}, \ n \in \mathbf{N}_+\right\}$.
   (f) The set of rational numbers.

2. Identify and draw surfaces given by the formulas below. When it's possible state their names and characteristic parameters. Otherwise, describe how the surfaces are constructed.

   (a) $f(x, y) = \sqrt{2 - x^2 - y^2 + 2x}$.
   (b) $f(x, y) = 2 - \sqrt{-3 - x^2 - y^2 - 4y}$.
   (c) $f(x, y) = -\sqrt{x^2 + y^2 + 2x + 1}$.
   (d) $f(x, y) = 3 + \frac{1}{2}\sqrt{x^2 + y^2 - 6x + 2y + 10}$.
   (e) $f(x, y) = \ln(x^2 + y^2 - 1)$.
   (f) $f(x, y) = (x^2 + y^2 + 2x)^2$.
   (g) $f(x, y) = 2x^2 - 3x + 1$.
   (h) $f(x, y) = \frac{2}{y}$.

3. Find and draw the implied domains of the following functions.

   (a) $f(x, y) = \sqrt{xy - 2y}$.
   (b) $f(x, y) = \frac{1}{x + y - 2} + \ln(2 - x^2)$.
   (c) $f(x, y) = \frac{\sqrt{4 - x^2 - y^2}}{\sqrt[4]{x^2 + y^2 + 4x}}$.

(d) $f(x,y,z) = \sqrt{x^2 + y^2 - 4z^2}$.

(e) $f(x,y,z) = \arccos(x^2 + y^2 + z^2 + 2z)$.

(f) $f(x,y,z) = \sqrt[3]{x} \cdot \dfrac{1}{y+3} \cdot \dfrac{1}{1-e^z}$.

4. Evaluate, exactly, the limits below.

(a) $$\lim_{\substack{x \to 5 \\ y \to 4 \\ z \to 1}} (2x - 3y + 2z) \sin\left(\frac{1}{x - y - z}\right).$$

(b) $$\lim_{\substack{x \to 1 \\ y \to -1}} (x^3 - y^3) \left\lfloor \frac{\pi}{x+y} \right\rfloor.$$

(c) $$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin(x+y)}{\sqrt[5]{x+y+1} - 1}.$$

(d) $$\lim_{\substack{x \to 2 \\ y \to 2 \\ z \to 0}} \frac{e^{x-y+z} + e^{y-x-z} - 2}{(x-y+z)^2}.$$

(e) $$\lim_{\substack{x \to -2 \\ y \to 1}} \frac{2^x \cdot 4^y - \cos(x+2y)}{x+2y}.$$

5. Show that the following limits don't exist.

(a) $$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\sin(xy)}{x^2 + y^2}.$$

(b) $$\lim_{\substack{x \to 0 \\ y \to 0 \\ z \to 0}} e^{\frac{1}{x-y-z}}.$$

(c) $$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{x + y - 3}{x^2 + y - 3}.$$

(d) $$\lim_{\substack{x \to 1 \\ y \to 1 \\ z \to 1}} (x + y - 2z) \ln\left((x-1)^2 + (y-1)^2 + (z-1)^2 \right).$$

6. Find all points at which the following functions are continuous.

(a) $f(x,y) = \begin{cases} x+y, \ x+y \geq 0, \\ 3x-5y+1, \ x+y<0. \end{cases}$

(b) $f(x,y) = \begin{cases} \sqrt{2-x^2-y^2}, \ x^2+y^2 \leq 1, \\ \sqrt{x^2+y^2}, \ x^2+y^2>1. \end{cases}$

(c) $f(x,y,z) = \operatorname{sgn}(x^2+y^4+(z+1)^6)$.

(d) $f(x,y,z) = \begin{cases} \frac{x+y+z}{x+y+2z}, \ x+y+2z \neq 0, \\ 1, \ x+y+2z=0. \end{cases}$

7. Find the value of a for which f are continuous on their domains.

(a) $f(x,y) = \begin{cases} x+y-1, \ y \geq x+1, \\ ax-2y+2, \ y<x+1. \end{cases}$

(b) $f(x,y) = \begin{cases} (x+ay)\ln(x+y), \ x+y>0, \\ 0, \ x+y \leq 0. \end{cases}$

(c) $f(x,y,z) = \begin{cases} \frac{x+y+az}{x+y+2z}, \ x+y+2z \neq 0, \\ 1, \ x+y+2z=0. \end{cases}$

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