5. Derivatives and differentiability of multivariable functions.

1. Calculate all second order partial derivatives of the functions below and verify validity of Schwarz's theorem.

   (a) $f(x, y) = (2x + y^2 + 3)^7$.
   (b) $f(x, y) = \sqrt{(x-1)^2 + y^2}$.
   (c) $f(x, y) = x \ln(xy^2 + y + 1)$.
   (d) $f(x, y, z) = \dfrac{1}{x^4 + y^4 + z^4}$.
   (e) $f(x, y, z) = \dfrac{x}{\sqrt[3]{y}} + \dfrac{y}{\sqrt[3]{z}} + \dfrac{z}{\sqrt[3]{x}}$.

2. Using Schwarz's theorem find all

   (a) third order partial derivatives of $f(x, y) = \cos(xy^2)$,
   (b) third order partial derivatives of $f(x, y, z) = x^2 \cdot \sqrt[4]{y} \cdot z$,
   (c) fourth order partial derivatives of $f(x, y) = \dfrac{y^3}{x^2}$,
   (d) nth order partial derivatives of $f(x, y) = e^{2x-y}$.

3. Verify that

   (a) $f(x, y) = \log(x^2 + y^2 - 2x - 2y + 2)$ is a solution to the equation $f_{xx} + f_{yy} = 0$,
   (b) $f(x, y, z) = \sqrt[3]{(2 + \sin(x - z)) \cdot (y - 2z)^4}$ is a solution to the equation $f_x + 2f_y + f_z = 0$.

4. Let $f(x, y) = \ln(2y + 8 - x^2 - y^2)$. Identify and draw the set of all points (x, y) for which $f_{xy}(x, y) \leq 0$.

5. (*) All nth order partial derivatives of some function f of k variables are to be calculated, on a given open subset of the domain of f. How many formulas, maximally, may we obtain if all these derivatives are continuous on this set?

6. Find an equation of the tangent plane to the graph of

   (a) $f(x, y) = \mathrm{tg}(x + 2y)$, at $P = (0, 0, f(0, 0))$,
   (b) $f(x, y) = \mathrm{arctg}(x^2 - y)$, at its intersection point with the Y-axis,
   (c) $f(x, y) = (2x + y - 1)^5$, at its intersection point with the line

   $$L : \begin{cases} x = 1 + t \\ y = -1 + 2t \\ z = -t \end{cases}, \ t \in \mathbf{R},$$

   (d) $f(x, y) = (x + 3y + 1)^4 + (x - y)^2$, at its common point with the XY-plane.

   Prove that these planes are really tangent planes by testing differentiability of these functions at suitable points.

7. A surface is represented by the equation $z = \sqrt{2x - y^2}$. Find its all points at which the tangent plane

(a) is perpendicular to the line $L : \begin{cases} x = 3t \\ y = -t \\ z = 1 - 2t \end{cases}, t \in \mathbf{R}$,

(b) is parallel to the plane $\Pi : -2x + 3z + 1 = 0$,

(c) is parallel to the plane $\Pi : x + 2y + 3z = 0$,

8. (*) The graph of $f = f(x, y)$ is a surface of revolution around the Z-axis. Assume that some neighbourhood of $P = (0, 0)$ is included in the domaln of f.

   (a) Show that a given point P either both derivatives f_x, f_y exist or none of them. In the first case find the values of these derivatives at P.

   (b) Show that existence of $f_x(P)$ implies differentiability of $f$at P and find the tangent plane to the graph of f at P.

9. Let $f(x, y) = |x| \cdot |y|$ and $\vec{v} = \left[-\frac{\sqrt{3}}{2}, \frac{1}{2}\right]$. Evaluate $\frac{\partial f}{\partial \vec{v}}(0, 2)$.

10. Evaluate directional derivatives of the functions below, at given points, in the directions of give vectors $\vec{v}$.

    (a) $f(x, y) = x^{xy}$, $P = (2, 1)$, $\vec{v} = \left[-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right]$.

    (b) $f(x, y) = \sin(2\pi \cos(x \operatorname{tg} y)))$, $P = \left(\frac{\pi}{3}, -\frac{\pi}{4}\right)$, $\vec{v} = \left[-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right]$.

    (c) $f(x, y, z) = (y + 2z)^x$, $P = (2, 2, 0)$, $\vec{v} = \left[-\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right]$.

11. Let $f(x, y) = \frac{3x + y}{x^2 + 1}$ and $P = (1, 1)$.

    (a) Find all unit vectors $\vec{v}$ along which $\frac{\partial f}{\partial \vec{v}}(P)$ attains its greatest value. Find this value.

    (b) Find all unit vectors $\vec{v}$ along which $\frac{\partial f}{\partial \vec{v}}(P)$ attains its least value. Find this value.

    (c) Find all unit vectors $\vec{v}$ along which $\frac{\partial f}{\partial \vec{v}}(P) = 0$.

    (d) For $\vec{v} = \left[\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right]$ idenfity and draw the set of all points (x, y) at which $\frac{\partial f}{\partial \vec{v}}(x, y) = 0$.

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