6. Applications of derivatives of multivariable functions.

1. Find the approximations of the numbers below, approximating an adequate function by its tangent plane. Compare the results with the exact values and evaluate relative errors of such approximations.

   (a) $\sin(0.1) \cdot \ln(0.95)$.

   (b) $\sqrt{1.03^2 + 2.02^2 + 1.95^2}$.

2. Approximating an adequate function by its tangent plane find, approximately, the answers to the following problems.

   (a) To calculate the average density of a given object its mass and volume were measured. The results are $530\text{g} \pm 1\text{g}$ and $13\text{cm}^3 \pm 1\text{cm}^3$. With what accuracy can the density be calculated?

   (b) Two sides of a triangle and the angle between them were measured. The results are $18\text{mm} \pm 1\text{mm}$, $7\text{mm} \pm 1\text{mm}$ and $30° \pm 1°$. With what accuracy can the area of the triangle be calculated?

   (c) Each of n numbers can be evaluated with a given accuracy $\delta > 0$. With what accuracy can their arithmetic mean be calculated?

   (d) Each of n positive numbers can be evaluated with a given accuracy $\delta > 0$. With what accuracy can their geometric mean be calculated?

3. Using basic definitions show that the following functions have extrema a given points. Verify which of them are global extrema.

   (a) $f(x,y) = -2(x-3)^6 + \dfrac{1}{1+|y+1|}$ at $P = (3,-1)$.

   (b) $g(x,y,z) = x^2 + \cos y + \sqrt[3]{z^2}$ at $P = (\pi, 0, 0)$.

   (c) $h(x,y) = x^4 - x^5 + y^2 + 2y$ at $P = (0,-1)$.

4. Using suitable logical operations and quantifiers state that a given functions does not have an extremum at $P \in \mathbf{R}^n$.

   Next, prove that the following functions don't have extrema at given points.

   (a) $f(x,y) = 2(x-3)^6 + \dfrac{1}{1+|y+1|}$ at $P = (3,-1)$,

   (b) $g(x,y,z) = x^2 + \cos y + \sqrt[3]{z^2}$ at $P = (0,0,0)$.

5. Find all local extrema of the following functions.

   (a) $f(x,y) = 2x^2 + 3xy + 5y^2 + 4x - 7y + 2$.

   (b) $f(x,y) = (y - 4x^2) \cdot (1-y)$.

   (c) $f(x,y) = (x^2 + y^2)e^{x-2y}$.

   (d) $f(x,y) = \dfrac{1-y}{\sqrt{x}} - \dfrac{x}{y}$.

   (e) $f(x,y) = \dfrac{4}{y} + \dfrac{y}{x} - x^2$.

(f) $f(x,y) = x^2 + y^2 - 2x + y - \ln y$.

6. (*) For two variables we define a polynomial of degree 2 by

   $W(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$,

   where $a, b, c, d, e, f \in \mathbf{R}$ and $a \neq 0 \vee c \neq 0$.

   Let use define a substitute of its discriminant by the same formula as for one variable, that is, $\Delta = b^2 - 4ac$.

   Investigate the extrema of W when $\Delta > 0$, and, next, when $\Delta < 0$.

7. Find all local extrema of the following functions, under given conditions.

   (a) $f(x,y) = \dfrac{x}{y+7}$, $x + \sqrt{y} = 0$.

   (b) $f(x,y) = xy + y$, $y = \sqrt{1-x^2}$.

   (c) $f(x,y) = y - \ln x$, $y = x^4$.

8. Find the greatest and the least values of the following functions defined on given sets D.

   (a) $f(x,y) = 2x^3 + 3y^4$, $D = \{(x,y) :\ x^4 + y^4 \leq 1\}$.

   (b) $f(x,y) = xy$, $D = \{(x,y) : 0 \leq x \leq \sqrt{4-y^2}\}$.

   (c) $f(x,y) = x^2 e^{-y^2}$, D is a triangle with vertices $(0,0)$, $(0,2)$ and $(2,2)$.

   (d) $f(x,y) = (x^2+y^2)e^x$, D is the square $[-1,1] \times [-1,1]$.

   (e) $f(x,y) = x^2 y$, D is a trapezium with vertices $(-1,0)$, $(-1,1)$, $(0,1)$ and $(1,0)$.

9. (a) Among all cuboids included in a ball of radius 1 find the dimensions of the one that has the largest possible volume.

   (b) A piece of rectangular metal sheet of length 5m and width 30cm is to be bent along two lines that are parallel to the longer side. In this way we construct a gutter whose cross section is an isosceles trapezium. If the gutter is to carry the maximum possible amount of water determine its shape and dimensions.

   (c) (*) Find the shortest distance betweeen the parabolas $y = x^2$ and $y = 3x^2 + 1$.

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