8. Applications of double integrals.

1. (a) Using double integrals evaluate the area of the region bounded by the graphs of the curves $y = \frac{1}{2}|x|$ and $y = 1 + \sqrt{1 - x^2}$.
   (b) Find this area directly, using geometry and formulas of areas of adequate figures.

2. Find the volumes of solids U given by the conditions below.
   (a) $U = \{(x, y, z) : 0 \le x \le 2,\ -1 \le y \le x,\ -x \le z \le 1 + xe^{2y}\}$.
   (b) $U = \left\{(x, y, z) : 0 \le 2y \le x \le \sqrt{1 - y^2},\ 0 \le z \le \dfrac{x}{x^2 + y^2 + 1}\right\}$.
   (c) U is bounded by the surfaces $x^2 + y^2 = 2Rx$, $z = 0$ and $z = a\sqrt{x^2 + y^2}$, where $R, a > 0$.

3. A sector of radius R and central angle α is rotated around its axis of symmetry and forms a solid of revolution. Using double integrals prove that the volume of the solid is given by the formula $V = \frac{2}{3}\pi R^3 \left(1 - \cos\frac{\alpha}{2}\right)$.

   Find the value of α for which the volume is a half of the volume of a semi-ball of radius R.

4. Find the areas of the surfaces that are graphs of the functions given below, over the indicated domains.
   (a) $f(x, y) = 2(x^2 + y^2)$, where $D_f = \{(x, y) : x^2 + y^2 \le 4x\}$.
   (b) $f(x, y) = Ax + By + C$, where D_f is any domain whose area is equal to S.
   (c) $f(x, y) = a\sqrt{(x - x_0)^2 + (y - y_0)^2}$, where D_f is any domain whose area is equal to S.

5. Using double integrals evaluate, approximately, the area of the whole part of the Arctic that lies within the Arctic Circle. Assume that the Earth is a ball with radius of length 6370km, and the latitude of the Arctic Circle is $66°40'$N.

   Give the answer in km^2, to 3 significant figures.

6. Evaluate the coordinates of the centres of mass of the following regions.
   (a) The set bounded by the axes and the graph of the curve $y = 2 - e^x$.
   (b) The set bounded by the graphs of the curves $y = x^2$ and $y = \sqrt{x}$.
   (c) The set bounded by the X-axis and the graph of the curve $y = a - x^p$, where $a, p > 0$.

7. Consider a sector of radius R and central angle α. Its axis of symmetry is the Y-axis and the vertex is located at the origin. Determine the coordinates of its centre of mass and, next, find the limimt position of this point when $\alpha \to (2\pi)^-$, and when $\alpha \to 0^+$. Comment on these results.

8. D is a set that can be represented as $D = D_1 \cup D_2 \cup ... \cup D_n$, where the sets D_1, D_2,...,D_n are regions whose interiors are mutually disjoint (so they may have common points only at their boundaries). For $i = 1, 2, ..., n$ let $S_i = (x_i, y_i)$ denote the centre of mass of D_i.

   Using the general formula of the coordinates of centre of mass show that the centre of mass of D is a point $S = (x_c, y_c)$ whose coordinates are weighted averages of adequate coordinates of S_i, and the weights are the areas of D_i. More precisely, we have

$$x_c = \sum_{i=1}^{n} \frac{|D_i|}{|D|} x_i = \frac{|D_1|}{|D|} x_1 + \frac{|D_2|}{|D|} x_2 + ... + \frac{|D_n|}{|D|} x_n,$$

$$y_c = \sum_{i=1}^{n} \frac{|D_i|}{|D|} y_i = \frac{|D_1|}{|D|} y_1 + \frac{|D_2|}{|D|} y_2 + ... + \frac{|D_n|}{|D|} y_n.$$

In vector form we get

$$\overrightarrow{OS} = \sum_{i=1}^{n} \frac{|D_i|}{|D|} \cdot \overrightarrow{OD_i} = \frac{|D_1|}{|D|} \cdot \overrightarrow{OD_1} + \frac{|D_2|}{|D|} \cdot \overrightarrow{OD_2} + ... + \frac{|D_n|}{|D|} \cdot \overrightarrow{OD_n}.$$

9. Using the formulas from the previous problem (so without any integration) and knowing positions of centres of mass of popular figures determine the centres of mass of the sets below.

   (a) A union of two discs with radii R_1 and R_2, respectively, and such that the distance d between their centres is at least $R_1 + R_2$.

   (b) A union of a right isosceless triangle with the shortest side of length a, and a semicircle with the diameter of length a, if the common part of these sets is the shortes side of the triangle.

   (c) A union of a square with sides of length a and an equilateral triangle with sides of length $a$if the common part of these sets is a side of the triangle.

   (d) A square with sides of length a from which a semicircle with the diameter of length a is removed.

10. Determine the moments of inertia of the following homogeneous figures of mass M.

    (a) A square of side a, about one of its sides.

    (b) A square of side a, about the line that joins the midpoints of two opposite sides,

    (c) A square of side a, about its diagonal.

    (d) A disc of radius R, about its diameter.

    (e) A disc of radius R, about a line that is tangent to the boundary circle.

    (f) A sector of radius R and central angle α, about its axis of symmetry.

    (g) A sector of radius R and central angle α, about its vertex.

    (h) An equilateral triangle of side a, about one of its sides.

    (i) An equilateral triangle of side a, about one of its heights.

Krzysztof „El Profe” Michalik