9. Laplace transform and differential equations.

1. Derive Laplace transforms of the functions given below.

   (a) $f(t) = C \cdot \mathbf{1}_{[a,b]}(t) = \begin{cases} C, & t \in [a,b], \\ 0, & t \notin [a,b], \end{cases}$ where $0 \le a < b$ and $C \neq 0$.

   (b) $f(t) = \cos(\beta t)$, $\beta \neq 0$.

   (c) $f(t) = \begin{cases} t, & t \in [0,a), \\ 2a - t, & t \notin [a, 2a], \\ 0, & t > a, \end{cases}$ where $a > 0$.

   (d) $f(t) = \mathbf{1}_A(t) = \begin{cases} 1, & t \in A, \\ 0, & t \notin A, \end{cases}$

   where $A = \bigcup_{n=0}^{\infty} [2n, 2n+1] = [0,1] \cup [2,3] \cup [4,5] \cup ... = \{t : \exists n \in \mathbf{N} \ \ t \in [2n, 2n+1]\}$.

2. Using basic properties of Laplace transform and formulas of the basic transforms identify Laplace transforms of the functions given below.

   (a) $f(t) = 2t^7 - 5t^4 + 7t^2 - 4 + t^3 \cdot e^{3t}$.

   (b) $f(t) = -5(t-2)^3 + 2^t$.

   (c) $f(t) = 5\cos\left(2t - \frac{\pi}{3}\right)$.

   (d) $f(t) = te^{-t}\cos(3t)$.

   (e) $f(t) = \begin{cases} (t-3)^4 \cdot e^{2(t-3)}, & t \ge 3, \\ 0, & t < 3. \end{cases}$

3. Find all continuous functions whose Laplace transforms are given below.

   (a) $\dfrac{1}{s-3} + \dfrac{5}{s^3} + \dfrac{2}{s^2+7} - \dfrac{s}{s^2+7}$.

   (b) $\dfrac{3s+2}{s^2 - s - 2}$.

   (c) $\dfrac{s^2+5}{s^3 - 3s + 2}$.

   (d) $\dfrac{s}{s^2 - 6s + 34}$.

   (e) $\dfrac{As+B}{(s-p)^2+q^2}$, where $q > 0$ and $A \neq 0 \vee B \neq 0$.

   (f) $\dfrac{e^{-2s}}{(s-2)^2}$.

   (g) $\dfrac{e^{-s}}{s^2+2s+2}$.

4. Using Laplace solve the following differential equations of variable $y = y(t)$.

   (a) $y' + y = t^2 e^{-t}$, $y(0) = 1$.

(b) $y' + y = 4\sin t,\ y(0) = -1.$

(c) $y' + y = 4\sin t,\ y\left(\frac{\pi}{2}\right) = -1.$

(d) $y' - \alpha y = k$, where $\alpha, k \in \mathbf{R}$.

(e) $y' - \alpha y = k,\ y(t_0) = y_0$, where $\alpha, k, t_0, y_0 \in \mathbf{R}$.

(f) $y'' - 3y' + 2y = 0,\ y(0) = 2, y'(0) = -1.$

(g) $y'' - 2py' + p^2 y = 0,\ y(t_0) = y_0, y'(t_0) = y_1$, where $p, t_0, y_0, y_1 \in \mathbf{R}$.

(h) $y'' + 4y = t,\ y(0) = 0, y'(0) = 1.$

(i) $y'' - 4y = t,\ y(0) = 0, y'(0) = 1.$

(j) $y'' - y = te^{2t},\ y(0) = y'(0) = 1.$

(k) $y'' - y = te^{t},\ y(0) = y'(0) = 1.$

(l) $y'' + 2y' + 2y = \sin(2t),\ y(0) = y'(0) = 0.$

(m) $y'' + 2y' + 2y = \cos t,\ y(0) = y'(0) = 0.$

(n) $y'' - 2y' + y = e^{t},\ y(0) = 2, y'(0) = 0.$

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