Functions and their properties

1. Find, using the exact values, the implied domains of the functions below

    1. $f(x) = \dfrac{2}{3x+1} + \dfrac{1}{(x-5)^2}$.
    2. $f(x) = \dfrac{3}{x^2-4} + \dfrac{1}{2x^2+x+1}$.
    3. $f(x) = \dfrac{1}{\frac{4}{x}+5} - \dfrac{2}{\frac{1}{x^2}-9}$.
    4. $f(x) = \dfrac{1}{8-2^x} + \dfrac{2}{(10^x-0.01)^2}$.
    5. $f(x) = \dfrac{1}{3^x+5} + \dfrac{1}{(0.4)^x-7}$.
    6. $f(x) = \sqrt{x^2+5x}$.
    7. $f(x) = \sqrt{-x^2-5x}$.
    8. $f(x) = \dfrac{1}{\sqrt{x^2-3}}$.
    9. $f(x) = -\dfrac{5}{\sqrt{3-x^2}} + \sqrt[3]{x}$.
    10. $f(x) = \sqrt[4]{x^2+x-2} + \sqrt{x+1}$.
    11. $f(x) = \sqrt{-x^2-2x+3} + \sqrt[6]{x}$.
    12. $f(x) = \sqrt{8-2^x} + \sqrt{10^x-0.01}$.
    13. $f(x) = \sqrt[4]{3^x+5} + \sqrt[4]{(0.4)^x-7}$.
    14. $f(x) = \sqrt{2x+1} + \dfrac{1}{x-3}$.
    15. $f(x) = \sqrt[4]{2-x} + \dfrac{1}{x^2} - 6\sqrt[5]{x+2}$.
    16. $f(x) = \sqrt{3x-1} + \dfrac{5}{x+3}$.
    17. $f(x) = \sqrt[8]{3-x} - \dfrac{1}{x^2-10}$.
    18. $f(x) = \sqrt{x^2+x+1} + \dfrac{1}{x^2+3}$.
    19. $f(x) = \sqrt{3-\sqrt[4]{x}}$.
    20. $f(x) = \sqrt[4]{2-\sqrt{x+1}}$.
    21. $f(x) = \sqrt[4]{\sqrt[4]{x+1}-2}$.
    22. $f(x) = \sqrt{\sqrt{8-x}-2}$.
    23. $f(x) = \sqrt{\sqrt{8-x}+2}$.
    24. $f(x) = \sqrt{\sqrt[3]{8-x}-2}$.
    25. $f(x) = \ln(x^2-5)$.
    26. $f(x) = \log_2(7-x^2)$.
    27. $f(x) = \log_{0.3}(x^2+x-2)$.
    28. $f(x) = \log_3\left((x+5)(x-4)\right)$.

29. $f(x) = \log_3(x+5) + \log_3(x-4)$.

30. $f(x) = \log_4(2^x - 4) + \ln(10000 - 10^x)$.

31. $f(x) = \log(3^x - 2) + \ln(4^x - 5)$.

32. $f(x) = \dfrac{1}{\log_3 x}$.

33. $f(x) = \dfrac{5}{3 - \log_2(x+1)}$.

34. $f(x) = \dfrac{1}{4 + \ln x}$.

35. $f(x) = \sqrt{\log_2 x + 5}$.

36. $f(x) = \sqrt[4]{3 - \ln x}$.

37. $f(x) = \sqrt{\log_{0.1}(x+5)}$.

38. $f(x) = \sqrt{\log_3(4-x) + 2}$.

39. $f(x) = \log(1 + \sqrt{x+3})$.

40. $f(x) = \log_7(1 - \sqrt{x+3})$.

41. $f(x) = \log_{0.4}(2 + \sqrt{2^x - 1})$.

42. $f(x) = \log_{0.4}(2 - \sqrt{2^x - 1})$.

43. $f(x) = \log_2(\log_3 x)$.

44. $f(x) = \ln(\log_{0.7} x)$.

45. $f(x) = \dfrac{x^2+3}{\log(1-\sqrt{x})}$.

46. $\sqrt{1 + 2\log_{0.01}(x+1)} + \dfrac{1}{x^2 - x - 6}$.

47. $f(x) = \ln\left(\dfrac{3}{\sqrt{2^x - 4} - 2}\right)$.

2. Draw a possible graph of f if f satisfies the conditions below.

1. $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to-2^-} f(x) = 3 = f(-2)$, $\lim_{x\to-2^+} f(x) = -\infty$, $\lim_{x\to 0} f(x) = 4$, $f(0) = 1$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = \infty$, $\lim_{x\to\infty} f(x)$ doesn't exist.
2. $\lim_{x\to-\infty} f(x) = -3$, $\lim_{x\to 1^-} f(x) = 2$, $\lim_{x\to 1^+} f(x) = 5$, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 1$, $\lim_{x\to 4^+} f(x)$ doesn't exist, $\lim_{x\to 5^-} f(x) = -\infty$, $\lim_{x\to 5^+} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$,
3. $\lim_{x\to-\infty} f(x)$ doesn't exist, $\lim_{x\to-3^-} f(x) = \infty$, $\lim_{x\to-3^+} f(x) = 0$, $f(-3) = 1$, $\lim_{x\to 1} f(x) = 2$, $f(1) = 0$, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 2$.
4. $\lim_{x\to-\infty} f(x) = -\infty$, $\lim_{x\to-1^-} f(x) = 4$, $\lim_{x\to-1^+} f(x) = 2 = f(-1)$, $\lim_{x\to 1^-} f(x)$ doesn't exist, $\lim_{x\to 2} f(x) = 3 = f(2)$, $\lim_{x\to 3^-} f(x) = \infty$, $\lim_{x\to 3^+} f(x) = -\infty$, $f(3) = 1$, $\lim_{x\to\infty} f(x) = \infty$,
5. $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to-4^-} f(x) = \lim_{x\to-4^+} f(x) = 2 = f(-4)$, $\lim_{x\to-2^-} f(x) = \infty$, $\lim_{x\to-2^+} f(x) = 0 = f(-2)$, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 2$, $f(1) = 1$, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = -\infty$, $\lim_{x\to\infty} f(x)$ doesn't exist.
6. $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to-1^-} f(x) = -1$, $\lim_{x\to-1^+} f(x) = 1$, $\lim_{x\to 0^+} f(x)$ doesn't exist, $f(0) = 0$, $\lim_{x\to 1} f(x) = 2 = f(1)$, $\lim_{x\to 2^-} f(x) = -\infty$, $\lim_{x\to 2^+} f(x) = \infty$, $\lim_{x\to\infty} f(x) = -\infty$,

3. Find the limits (one-sided or double-sided) at the ends of the domains of all functions f in task 1. Hence, state the equations of all horizontal and all vertical asymptotes of f. Sketch the graphs of f, clearly indicating your findings.

4. Show that

   1. $y = x$ is an asymptote of $f(x) = x + \dfrac{1}{(x-2)^2}$ both at ∞ and at $-\infty$,
   2. $y = -x + 2$ is an asymptote of $f(x) = -x + 2 - \dfrac{7}{x^2+1}$ both at ∞ and at $-\infty$,
   3. $y = 2x + 3$ is an asymptote of $f(x) = \dfrac{2x^2+5x+1}{x+1}$ both at ∞ and at $-\infty$,
   4. $y = 3x - 1$ is an asymptote of $f(x) = \dfrac{6x^3-2x^2-3x}{2x^2-1}$ both at ∞ and at $-\infty$,
   5. $y = x + 7$ is an asymptote of $f(x) = x + 7 + 2^x$ at $-\infty$ but not at ∞,
   6. $y = 4x + 2$ is an asymptote of $f(x) = 4x + \sqrt{(1.01)^x + 4}$ at $-\infty$ but not at ∞,
   7. $y = -2x + 1$ is an asymptote of $f(x) = -2x + 1 - 7.8 \cdot \left(\dfrac{2}{3}\right)^x$ at ∞ but not at $-\infty$.
   8. $y = -2x + 1$ is an asymptote of $f(x) = -2x + \sqrt{(\ln 2)^x + 1}$ at ∞ but not at $-\infty$.
   9. $y = -3x - 2$ is an asymptote of $f(x) = \dfrac{5}{2^x+3^x} - 3x - 2$ at ∞ but not at $-\infty$.

5. Find all asymptotes of the functions below.

   1. $f(x) = 3x + 5 - \dfrac{1}{x-7}$.
   2. $f(x) = \dfrac{1}{\sqrt{x+1}} - 3x$.
   3. $f(x) = x + \dfrac{1}{3^x+2}$.
   4. $f(x) = -2x + \dfrac{5}{2^x-8}$.
   5. $f(x) = x + |x| + \dfrac{1}{x}$.
   6. $f(x) = |2x+3| - 2x + \dfrac{1}{x^2+2x+5}$.

6. Investigate whether the functions below are continuous at given points. In case of discontinuity determine its type (jump, gap, asymptote, non-existing one-sided limit). Sketch the graphs of f, clearly indicating your findings.

   1. $f(x) = \begin{cases} \sqrt{x+2} & , x \geq 1, \\ 3^{\frac{x}{3-x}} & , x < 1, \end{cases}$ at $x = 1$.
   2. $f(x) = \begin{cases} \dfrac{1}{\ln x} & , 0 < x < 1, \\ 1, & x = 0, \\ \log_2(1+x) & , -1 < x < 0, \end{cases}$ at $x = 0$.
   3. $f(x) = \begin{cases} x^2 + 3x + 5 & , x > -2, \\ 2^x & , x \leq -2, \end{cases}$ at $x = -2$.
   4. $f(x) = \begin{cases} 3^{\frac{1}{x}} & , x \neq 0, \\ 1 & , x = 0, \end{cases}$ at $x = 0$.

5. $f(x) = \begin{cases} 3^{\frac{1}{x^2}} & , x \neq 0, \\ 1 & , x = 0, \end{cases}$ at $x = 0$.

6. $f(x) = \begin{cases} 3^{-\frac{1}{x^2}} & , x \neq 0, \\ 1 & , x = 0, \end{cases}$ at $x = 0$.

7. $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & , x \neq 0, \\ 1 & , x = 0, \end{cases}$ at $x = 0$.

8. $f(x) = \begin{cases} \log_3(x+3) - \log_3(x^2+17) & , x \geq -1, \\ -\sqrt{3x^2+1} & , x < -1, \end{cases}$ at $x = -1$.

9. $f(x) = \begin{cases} \dfrac{1}{1+3^{\frac{1}{x}}} & , x \neq 0, \\ 0 & , x = 0, \end{cases}$ at $x = 0$.

10. $f(x) = \begin{cases} 3 + \cos\left(\frac{1}{x^4}\right) & , x \neq 0, \\ 1 & , x = 0, \end{cases}$ at $x = 0$.

7. Find all $A,\ B \in \mathbf{R}$ for which the following functions are continuous.

1. $f(x) = \begin{cases} A + 3^x & , x \geq 1, \\ \sqrt[3]{x+7} & , x < 1. \end{cases}$

2. $f(x) = \begin{cases} \dfrac{2^x - A}{1+3^x} & , x \geq 0, \\ A\log_2(8-x) & , x < 0. \end{cases}$

3. $f(x) = \begin{cases} \sqrt{x} & , x \geq 4, \\ Ax + B & , 0 < x < 4, \\ x - 1 & , x \leq 0. \end{cases}$

4. $f(x) = \begin{cases} \log_9(2x-1) & , x > 2, \\ Ax^2 + B & , -1 \leq x \leq 2, \\ 3^x & , x < -1. \end{cases}$

5. $f(x) = \begin{cases} \dfrac{1}{\ln|x|} & , x \neq 0, \\ A & , x = 0. \end{cases}$

6. $f(x) = \begin{cases} \dfrac{1}{1+3^{-\frac{1}{x^2}}} & , x \neq 0, \\ A & , x = 0. \end{cases}$

8. Let $a > 0$ and $a \neq 1$. Check which of the functions below are even, odd, or neither even nor odd.

1. $f(x) = 3x^2 + 5|x| - 1.$
2. $f(x) = \sqrt[3]{x^4} - 2.$
3. $f(x) = \sqrt[3]{x^5} - 2.$
4. $f(x) = 2x^5 + x^3 - \dfrac{4}{x}.$
5. $f(x) = 2x^5 + x^3 - \dfrac{4}{x} + 1.$
6. $f(x) = \dfrac{1}{x^3} - \dfrac{6}{x^5}.$

7. $f(x) = \sqrt{1-x} + \sqrt{1+x}$.

8. $f(x) = \dfrac{1}{\log_a(1-x)} - \dfrac{1}{\log_a(1+x)}$.

9. $f(x) = \dfrac{1}{x\log_a(1-x)} - \dfrac{1}{x\log_a(1+x)}$.

10. $f(x) = \dfrac{a^x+1}{a^x-1}$.