Math-algebra. Complex numbers

1. Simplify the following expressions and give the final answer in cartesian form.

   (a) $\dfrac{(1+2i)^2}{2+i}$

   (b) $(2-i)^3 + (2+i)^3$

   (c) i^n, $n \in \mathbf{N}$

   (d) $i + i^2 + i^3 + ... + i^{50}$

   (e) $Re(1-5i)^4$

   (f) $Im\left(\dfrac{i}{1+i}\right)$

2. Find the modulus of the numbers below.

   (a) $-5 + 12i$

   (b) $\dfrac{1-2i}{3+i}$

   (c) $(1+i)^{110}$

   (d) $z + 1 - 3i$, where $z = x + yi$, $x, y \in \mathbf{R}$

3. Prove that the following formulas are true for all complex z and w.

   (a) $Re(z+w) = Re(z) + Re(w)$, $Re(z-w) = Re(z) - Re(w)$, $Im(z+w) = Im(z) + Im(w)$ and $Im(z-w) = Im(z) - Im(w)$.

   (b) $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{z-w} = \overline{z} - \overline{w}$, $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$, and if $w \neq 0$ then $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$. Also, deduce for $n \in \mathbf{N}_+$ that $\overline{z^n} = (\overline{z})^n$.

   (c) $|z \cdot w| = |z| \cdot |w|$, and if $w \neq 0$ then $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$. Also, deduce for $n \in \mathbf{N}_+$ that $|z^n| = |z|^n$.

   (d) $|z+w| \leq |z| + |w|$.

   (e) $|z+w| = |z| + |w| \Leftrightarrow z = 0 \vee w = 0 \vee \dfrac{z}{w} > 0$.

4. Solve the following complex equations and inequalities. In case of infinitely many solutions plot them in the complex plane.

   (a) $\overline{z} + 2 - i = (3+2i)z$.

   (b) $Re(2iz) = 4Im((1-i)\overline{z})$.

   (c) $Re(z^2) \leq 0$.

   (d) $Im(z^3) = 0$.

   (e) $Re\left(\dfrac{2}{z}\right) = 0$.

   (f) $Im\left(\dfrac{2}{z}\right) = 1$.

   (g) $Im\left(\dfrac{1+2i}{z+1}\right) \geq 0$.

   (h) $|z + 5 - 3i| = 2$.

(i) $|iz + 3 + 2i| > 1$.

(j) $|z + 1 - i| = |z + 4|$.

(k) $|i\overline{z} - 2 + 3i| \leq |z - 3|$.

(l) $2|z + 3| = |z + 6i|$.

(m) $2|z + 1| \leq |\overline{z} + 10 + 12i|$.

5. Represent the numbers below in polar and trigonometric form.

(a) $3 + 3i$

(b) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(c) $-1 - i\sqrt{3}$

(d) $\sqrt{5} - i\sqrt{5}$

(e) $z \in \mathbf{R}$, $z \notin 0$

(f) $z = yi$, $y \in \mathbf{R}$, $y \notin 0$

6. Find the polar and the cartesian form of the numbers below.

(a) $(-1 + i\sqrt{3})^{14}$

(b) $(1 - i)^{-23}$

(c) $\frac{(1 + i)^{40}}{1 - i}$

(d) $\frac{\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{33}}{(\sqrt{3} - i)^7}$

7. (Task 4 continued). Solve the following complex equations and inequalities. In case of infinitely many solutions plot them in the complex plane.

(a) $Re\left(z(1 + i\sqrt{3})^8\right) = 128$.

(b) $Im\left((\overline{z} + 1)(-\sqrt{3} + i)^9\right) < 256$.

(c) $\left|z\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{15} + 2\right| \leq 5$.

(d) $arg(z + 2 - 3i) = \frac{\pi}{4}$.

(e) $\frac{\pi}{3} < arg(z - 2i) \leq \frac{5\pi}{6}$.

(f) $arg(iz) = \frac{\pi}{3}$.

(g) $\frac{\pi}{4} \leq arg\left((1 + i)(z + 2)\right) \leq \frac{5\pi}{4}$.

(h) $arg\left(\frac{-1 + i}{z}\right) = \frac{\pi}{4}$.

(i) $\begin{cases} |z - 3i| = 1, \\ arg(z) = \frac{\pi}{2}. \end{cases}$

(j) $\begin{cases} |z-1| = 3, \\ arg(z+i) = \dfrac{\pi}{4}. \end{cases}$

(k) $\begin{cases} |z+1| = 5, \\ arg(z+3-3i) = \dfrac{3\pi}{4}. \end{cases}$

(l) $\begin{cases} arg(z+7) = -\dfrac{\pi}{2}, \\ arg(z+5-i) = \dfrac{4\pi}{3}. \end{cases}$

(m) $\begin{cases} arg(z-2i) = \dfrac{\pi}{6}, \\ arg(z-1-3i) = \dfrac{\pi}{3}. \end{cases}$

8. Using geometrical interpretation of complex numbers, or otherwise, show that if $z,\ w \in \mathbf{C}$ and $|z| = |w|$ then $arg(z+w) = \dfrac{arg(z) + arg(w)}{2}$. Using this formula find the polar form of

   (a) $2 + \sqrt{3} + i$,

   (b) $-\sqrt{2} + 1 + i$,

   (c) $\sqrt{2} + 1 + i(\sqrt{2} + \sqrt{3})$.

9. Find the following roots of complex numbers. State the answers both in polar and cartesian form.

   (a) $\sqrt{-3+4i}$

   (b) $\sqrt{7+i}$

   (c) $\sqrt[3]{8i}$

   (d) $\sqrt[3]{-27}$

   (e) $\sqrt[4]{-1}$

   (f) $\sqrt[4]{-1+i\sqrt{3}}$

   (g) $\sqrt[5]{1}$

   (h) $\sqrt[4]{i}$

   (i) $\sqrt[3]{1+i}$

   (j) $\sqrt[4]{(1+2i)^4}$

   (k) $\sqrt[3]{(1+2i)^6}$

   (l) $\sqrt[n]{z^n}$, $n \in \mathbf{N}_+$

10. Verify that if $z < 0$ then $\sqrt{z} = \pm i\sqrt{|z|}$ and for other complex z the (complex) square root of z is $\sqrt{z} = \pm\sqrt{|z|} \cdot \dfrac{z + |z|}{|z + |z||}$.

11. Using complex roots solve the equations below.

    (a) $az^2 + bz + c = 0$, $a, b, c \in \mathbf{R}$, $a \neq 0$, $b^2 - 4ac < 0$

    (b) $z^2 + (1-i)z - 2 - i = 0$

    (c) $z^3 = (2z+i)^3$

    (d) $z^4 = -4(z+1)^4$

12. Let z, $z_0 \in \mathbf{C}$, $\alpha \in \mathbf{R}$. Explain why the transformations f, g such that $w = f(z) = ze^{i\alpha}$ and $w = g(z) = z_0 + (z - z_0)e^{i\alpha}$ represent, geometrically, rotations of points (vectors) in the XY plane. Find the parameters of these rotations.

    Using this method find the image of $P = (1, 3)$ when P is rotated

    (a) around the origin by $\alpha = 30°$ anticlockwise,

    (b) around $Q = (2, -1)$ by $\alpha = 135°$ clockwise.

13. Using complex numbers derive the formula of $\sin(5x)$ in terms of $\sin x$ only.

14. Using complex numbers find a formula of

    (a) $\cos x + \cos(2x) + ... + \cos(nx)$,

    (b) $2 \sin x + 4 \sin(2x) + ... + 2^n \sin(nx)$.

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