Math-algebra. Polynomials and rational functions

1. Divide P by Q if
   (a) $P(x) = 3x^6 + x^3 + x + 1$, $Q(x) = x^3 - x + 1$,
   (b) $P(x) = x^2 + 5x + 1$, $Q(x) = -2x^2 + 5$,
   (c) $P(x) = 2x^3 + x^2 + x - 7$, $Q(x) = 3x + 5$,
   (d) $P(x) = x^3 - 3x + 2$, $Q(x) = x + 2$.
2. (a) $P(x) = 4x^{211} + cx^3 - x^2 + 1$ is divisible by $Q(x) = x + 1$. Find c.
   (b) $P(x) = ax^3+bx^2+x-2$ has a root $x = 1$ and leaves a reminder -5 when divided by $Q(x) = x-2$. Find a and b.
3. The following polynomials have integer repeated roots. Find their multiplicity.
   (a) $P(x) = x^5 - 4x^4 + 3x^3 + 5x^2 - 8x + 4$
   (b) $P(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$
   (c) $P(x) = x^6 - 4x^5 + 7x^4 - 8x^3 + 7x^2 - 4x + 1$
   (d) $P(x) = x^7 + 3x^6 + 3x^5 + 2x^4 + 5x^3 + 9x^2 + 7x + 2$
4. Factorize the polynomials below over the
   - real number field,
   - complex number field.

   Use any reasonable methods: finding integer/rational/complex roots, grouping suitable terms, substituting a new variable etc.

   (a) $x^3 - 19x + 30$
   (b) $2x^3 + 3x^2 + 2x + 1$
   (c) $-\frac{1}{2}x^3 + x - \frac{1}{2}$
   (d) $5x^3 - 24x^2 + 36x - 16$
   (e) $\frac{1}{9}x^3 - x^2 + 3x - 3$
   (f) $x^4 - 3x^3 - 4x^2 + 18x - 12$
   (g) $x^4 + x^3 - x - 1$
   (h) $-9x^4 - 3x^3 + 23x^2 - 13x + 2$
   (i) $8x^4 + 20x^3 - 42x^2 + 23x - 4$
   (j) $16x^4 + 32x^3 + 24x^2 + 8x + 1$
   (k) $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$
   (l) $x^5 - x^4 - 2x^3 + 5x^2 - 5x + 2$
   (m) $x^6 + 5x^5 + 7x^4 - 2x^3 - 13x^2 - 11x - 3$
   (n) ($\star$) $x^5 + 32$
   (o) ($\star$) $x^7 - 1$

5. Factorize the following quartic polynomials over the real number field.

   (a) $x^4 - 3x^2 + 2$
   (b) $x^4 + x^2 - 2$
   (c) $2x^4 + 3x^2 + 1$
   (d) $x^4 - 6x^2 + 9$
   (e) $\frac{1}{2}x^4 + x^2 + \frac{1}{2}$
   (f) $x^4 + 9$
   (g) $x^4 - x^2 + 25$
   (h) $3x^4 + 5x^2 + 9$

6. $x = -2 + i$ is a root of $P(x) = -x^4 - 3x^3 - 3x^2 - 3x - 10$. Factorize P over the

   - real number field,
   - complex number field.

7. $x = i\sqrt{3}$ is a double root of $P(x) = 2x^6 + x^5 + 13x^4 + 6x^3 + 24x^2 + 9x + 9$. Factorize P over the

   - real number field,
   - complex number field.

8. Using the formula of $\sin(5x)$ in terms of $\sin x$ only (List 1, task 13) find the exact values of $\sin\left(\frac{\pi}{5}\right)$ and $\cos\left(\frac{\pi}{5}\right)$.

9. Find the remainder when $P(x) = 2x^{151} + 4x - 1$ is divided by

   (a) $Q(x) = x^2 - x$,
   (b) $Q(x) = x^3 - x$,
   (c) $Q(x) = x^2 + 1$,
   (d) $Q(x) = x^2 + 2x + 2$,
   (e) (*) $Q(x) = x^2 - 2x + 1$,
   (f) (*) $Q(x) = x^4 - 3x^3 + 3x^2 - x$.

10. Decompose the functions below into real partial fractions.

    (a) $f(x) = \dfrac{2x+1}{x^2 - 4x + 3}$
    (b) $f(x) = \dfrac{x^2}{x^3 + 2x^2 - x - 2}$
    (c) $f(x) = \dfrac{5}{x^3 - 3x + 2}$
    (d) $f(x) = \dfrac{-x^2 - 4x - 3}{x^3 + x + 2}$
    (e) $f(x) = \dfrac{-4x^2 + 26x - 34}{x^4 - 8x^3 + 18x^2 - 27}$
    (f) $f(x) = \dfrac{2x^3 + 4x^2 + 6x + 9}{x^4 + 5x^2 + 6}$
    (g) $f(x) = \dfrac{x^3 + 2x + 1}{x^4 + 2x^2 + 1}$

11. Represent the functions below as a sum of a polynomial and real partial fractions.

(a) $f(x) = \dfrac{5x^5 + x - 1}{x^3 + x^2}$

(b) $f(x) = \dfrac{4x^2 + x - 1}{x^2 - 3x + 2}$

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