Math-algebra. Matrices and determinants

1. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 5 & -2 \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$.

   Find, when possible, all products of the form XY and XY^T, where $X, Y \in \{A, B, C, D, E\}$.

2. For all matrices below

   - guess the formulas of their nth powers,
   - (*) prove these formulas by induction.

   (a) $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$,

   (b) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$,

   (c) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ (matrix of rotation around $(0,0)$ through the angle of α),

   (d) $\begin{bmatrix} \frac{1-a^2}{1+a^2} & \frac{2a}{1+a^2} \\ \frac{2a}{1+a^2} & -\frac{1-a^2}{1+a^2} \end{bmatrix}$ (matrix of symmetry about the line $y = ax$),

   (e) $\begin{bmatrix} \frac{1}{1+a^2} & \frac{a}{1+a^2} \\ \frac{a}{1+a^2} & \frac{a^2}{1+a^2} \end{bmatrix}$ (matrix of perpendicular projection onto the line $y = ax$),

   (f) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

   (g) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,

   (h) (*) $\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & a_3 & \dots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_n \end{bmatrix}$.

3. Solve the following matrix equations.

   (a) $\begin{bmatrix} 3 & 5 & 7 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix}$,

   (b) $X \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}^T - \begin{bmatrix} 6 & 10 & 14 \end{bmatrix}$,

   (c) $X^T = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} X$,

(d) $X^2 = \begin{bmatrix} 4 & 0 \\ -1 & 1 \end{bmatrix}$,

(e) $X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

(f) (*) $aX + bX^T = I_{n\times n},\ a, b \neq 0$.

4. (a) Prove that if $A = [a_{ij}]_{n\times n}$ and $B = [b_{ij}]_{n\times n}$ are diagonal matrices of the same dimensions then AB is also diagonal. Furthermore, the diagonal of AB is $\{a_{11}b_{11}, a_{22}b_{22}, ..., a_{nn}b_{nn}\}$.

   (b) Prove that an analogous theorem is true for upper triangle and lower triangle matrices.

5. Show that

   (a) if A is a square matrix then $A + A^T$ is symmetric while $A - A^T$ and $A^T - A$ are antisymmetric,

   (b) if B is any matrix then BB^T and B^TB are symmetric.

6. Show that if A and B are commutative then

   (a) they are square of the same dimensions,

   (b) $ABBA = BABA = A^2B^2$,

   (c) $(A \pm B)^2 = A^2 \pm 2AB + B^2$,

   (d) $(A \pm B)^3 = A^3 \pm 3A^2B + 3AB^2 \pm B^3$,

   (e) $A^2 - B^2 = (A - B)(A + B)$,

   (f) $A^3 \pm B^3 = (A \pm B)(A^2 \mp AB + B^2)$.

7. Prove that if a matrix B can be represented as a polynomial of another matrix A, that is, $B = c_0I + c_1A + c_2A^2 + ... + c_kA^k$ for some numbers $c_0, c_1, ..., c_n$ and $k \in \mathbf{N}^+$, then A and B are commutative matrices. Using this result show that commutative are

$$A = \begin{bmatrix} 2 & 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & -1 & -2 & 4 & 2 \\ 1 & 1 & 4 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 & -6 & -7 \\ -1 & -3 & -5 & -7 & 6 & -9 \\ 1 & -1 & 1 & -1 & 1 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & 4 & 6 & 8 & 10 \\ 2 & 3 & -2 & -4 & 8 & 4 \\ 2 & 2 & 5 & 2 & 2 & 2 \\ 2 & 4 & 6 & 7 & -12 & -14 \\ -2 & -6 & -10 & -14 & 9 & -18 \\ 2 & -2 & 2 & -2 & 2 & 11 \end{bmatrix}.$$

8. (*) Find all matrices that are commutative with **every** 2×2 matrix .

9. We know that, for numbers, $x^2 = 0 \Leftrightarrow x = 0$, $x^2 = 1 \Leftrightarrow x = \pm 1$ and $xy = 0 \Leftrightarrow x = 0 \vee y = 0$. Show that analogous equations for square matrices are false. In other words, find X, Y such that

   (a) $X^2 = \mathbf{0}$ but $X \neq \mathbf{0}$,

   (b) $X^2 = I$ but $X \neq I$ and $X \neq -I$

   (c) $X \neq Y$ and $XY = \mathbf{0}$ but $X \neq \mathbf{0}$ and $Y \neq \mathbf{0}$.

10. Find the determinants of the following matrices.

   (a) $\begin{bmatrix} 1 & 5 \\ 6 & 1 \end{bmatrix}$,

   (b) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$,

(c) $\begin{bmatrix} \dfrac{1-a^2}{1+a^2} & \dfrac{2a}{1+a^2} \\ \dfrac{2a}{1+a^2} & -\dfrac{1-a^2}{1+a^2} \end{bmatrix}$,

(d) $\begin{bmatrix} \dfrac{1}{1+a^2} & \dfrac{a}{1+a^2} \\ \dfrac{a}{1+a^2} & \dfrac{a^2}{1+a^2} \end{bmatrix}$,

(e) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

(f) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & a \end{bmatrix}$,

(g) $\begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} - aI_{3\times 3}$.

11. Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 5 & 3 \\ 7 & 0 & -1 & -2 \end{bmatrix}$. Expand $\det(A)$ along

    (a) the first row,

    (b) the second column.

    Then evaluate $\det(A)$.

12. Prove that the determinant of a triangle matrix is the product of the elements from its diagonal.

13. Calculate the following determinants

    (a) $\begin{vmatrix} 1 & 5 & 6 & -3 & 4 \\ 1 & 4 & 7 & 8 & -5 \\ 2 & 5 & -1 & 3 & 3 \\ -2 & -10 & 7 & 1 & 6 \\ -1 & 3 & 7 & -2 & 4 \end{vmatrix}$,

    (b) $\begin{vmatrix} 6 & 3 & 2 & -1 & -8 \\ 2 & 1 & 0 & 0 & -5 \\ -4 & 2 & 7 & 3 & 8 \\ 5 & 2 & 9 & 6 & 7 \\ -2 & 3 & 7 & -2 & a \end{vmatrix}$,

    (c) $\begin{vmatrix} a & a & a & \dots & a & a \\ b & a & a & \dots & a & a \\ b & b & a & \dots & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \dots & a & a \\ b & b & b & \dots & b & a \end{vmatrix}_{n\times n}$,

(d) (⋆) $$\begin{vmatrix} a & b & 0 & 0 & \dots & 0 & 0 \\ 0 & a & b & 0 & \dots & 0 & 0 \\ 0 & 0 & a & b & \dots & 0 & 0 \\ 0 & 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a & b \\ b & 0 & 0 & 0 & \dots & 0 & a \end{vmatrix}_{n\times n},$$

(e) (⋆) $$\begin{vmatrix} a & b & b & \dots & b & b \\ b & a & b & \dots & b & b \\ b & b & a & \dots & b & b \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b & b & b & \dots & a & b \\ b & b & b & \dots & b & a \end{vmatrix}_{n\times n}.$$

14. Find all parameters for which the following matrices are singular.

(a) $\begin{bmatrix} 1 & 2 \\ p & p^3 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & \cos\alpha \\ \cos\alpha & \cos(2\alpha) \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & 0 & x \\ 0 & x & 0 \\ x & 0 & 1 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & a \end{bmatrix}$,

(e) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \lambda I_{3\times 3}$.

15. If $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = x$ find the value of

(a) $\begin{vmatrix} a_4 & b_4 & c_4 & d_4 \\ a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \end{vmatrix}$,

(b) $\begin{vmatrix} d_4 & d_3 & d_2 & d_1 \\ c_4 & c_3 & c_2 & c_1 \\ b_4 & b_3 & b_2 & b_1 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix}$,

(c) $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ 2a_2 & 2b_2 & 2c_2 & 2d_2 \\ 3a_3 & 3b_3 & 3c_3 & 3d_3 \\ 4a_4 & 4b_4 & 4c_4 & 4d_4 \end{vmatrix}$,

(d) $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_1+a_2 & b_1+b_2 & c_1+c_2 & d_1+d_2 \\ a_1+a_2+a_3 & b_1+b_2+b_3 & c_1+c_2+c_3 & d_1+d_2+d_3 \\ a_1+a_2+a_3+a_4 & b_1+b_2+b_3+b_4 & c_1+c_2+c_3c_3+c_4 & d_1+d_2+d_3+d_4 \end{vmatrix}$,

(e) $\begin{vmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 & d_1+d_2 \\ a_2+a_3 & b_2+b_3 & c_2+c_3 & d_2+d_3 \\ a_3+a_4 & b_3+b_4 & c_3+c_4 & d_3+d_4 \\ a_4+a_1 & b_4+b_1 & c_4+c_1 & d_4+d_1 \end{vmatrix}$.

16. If $A = A_{5\times 5}$ with $\det(A) = 2$ find the value of

(a) $\det(2A)$,
(b) $\det(AA^TA)$,
(c) $\det(-3A^2)$,
(d) $\det\left(A^m\left(A^T\right)^n\right)$, $m, n \in \mathbf{N}^+$,

17. Find $\det(A)$ if $\det(2A^2) = 144$ and $\det(3A^T) = 243$.

18. Find the rank of the following matrices.

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

(b) $\begin{bmatrix} 2 & 5 & 7 & 6 \\ 1 & 3 & 4 & 5 \\ 8 & -7 & 1 & -2 \end{bmatrix}$,

(c) $\begin{bmatrix} 2 & 5 & 7 & 1 & 8 \\ 0 & 1 & 3 & 4 & 1 \\ 2 & 3 & 1 & -7 & 6 \end{bmatrix}$,

19. If $a \in \mathbf{R}$, find the rank of the following matrices.

(a) $\begin{bmatrix} 1 & a & 2 \\ 2 & 3 & 7 \\ 1 & 0 & 1 \end{bmatrix}$,

(b) $\begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$,

(c) $\begin{bmatrix} 1 & a & a & a \\ 1 & 1 & a & a \\ 1 & 1 & 1 & a \end{bmatrix}$,

(d) $\begin{bmatrix} 2 & 5 & 7 & a & 8 \\ 0 & 1 & 3 & 4 & a \\ 2 & 3 & 1 & -7 & 6 \end{bmatrix}$,

20. For all matrices given below find the conditions under which they are invertible. Then find the inverse to these matrices using the explicit formula.

(a) $\begin{bmatrix} 1 & 5 \\ 6 & p \end{bmatrix}$,

(b) $\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$,

(c) $\begin{bmatrix} \dfrac{1-a^2}{1+a^2} & \dfrac{2a}{1+a^2} \\ \dfrac{2a}{1+a^2} & -\dfrac{1-a^2}{1+a^2} \end{bmatrix}$,

(d) $\begin{bmatrix} \dfrac{1}{1+a^2} & \dfrac{a}{1+a^2} \\ \dfrac{a}{1+a^2} & \dfrac{a^2}{1+a^2} \end{bmatrix}$,

(e) $\begin{bmatrix} 1 & 0 & x \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$,

(f) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & x \end{bmatrix}$,

(g) $\begin{bmatrix} 2 & 3 & 5 \\ 4 & -1 & 2 \\ x & 1 & 1 \end{bmatrix}$,

(h) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & -1 & 0 \\ p & 1 & 1 & 1 \end{bmatrix}$.

21. By method of elimination find the inverse matrices to

(a) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$,

(b) $\begin{bmatrix} 1 & 0 & x \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$,

(c) $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & -1 & 0 \\ p & 1 & 1 & 1 \end{bmatrix}$.

22. Let $A \neq \mathbf{0}$ be a matrix such that $A^k = \mathbf{0}$ for some $k \in \mathbf{N}^+$. Show that the inverse to $I - A$ is $(I-A)^{-1} = I + A + A^2 + A^3 + ... + A^{k-1}$.

23. (a) Prove that if $A = [a_{ij}]_{n \times n}$ is invertible and diagonal then A^{-1} is also diagonal. Furthermore, the diagonal of A^{-1} is $\left\{ \dfrac{1}{a_{11}}, \dfrac{1}{a_{22}}, ..., \dfrac{1}{a_{nn}} \right\}$.

(b) Prove that an analogous theorem is true for upper triangle and lower triangle matrices.

24. Using suitable inverse matrices solve the following matrix equations.

(a) $\begin{bmatrix} 3 & 5 & 7 \\ 0 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix}$,

(b) $X \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$,

(c) $X \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & 1 & 2 & -3 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$,

(d) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} X \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -3 & 0 \end{bmatrix}$,

(e) $\begin{bmatrix} \frac{1-a^2}{1+a^2} & \frac{2a}{1+a^2} \\ \frac{2a}{1+a^2} & -\frac{1-a^2}{1+a^2} \end{bmatrix} X \begin{bmatrix} a & -1 \\ 1 & a \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} X + 2X = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$,

(g) $X(I-X)^{-1} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

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