Math-algebra. Vector geometry

1. Using vectors show that a quadrilateral whose diagonals bisect each other is a parallelogram.

2. Let $\vec{u}$ be the position vector of point A and $\vec{v}$ be the position vector of point B. Consider a point that divides the line segment AB in the ratio $p : q$, with $p, q > 0$. That is, such P from this line segment for which $\frac{|AP|}{|BP|} = \frac{p}{q}$. Find the position vector of P.

3. Show that the midpoints of the sides of any quadrilateral are vertices of some parallelogram.

4. Investigate which $\vec{u}$ and $\vec{v}$ listed below are

   - parallel,
   - perpendicular.

   (a) $\vec{u} = [12, -15]$ and $\vec{v} = [-8, 10]$.
   (b) $\vec{u} = [1, a]$ and $\vec{v} = [-a, 1]$, $a \in \mathbf{R}$.
   (c) $\vec{u} = [e, e^2, e^3]$ and $\vec{v} = [e^4, e^5, e^6]$.
   (d) $\vec{u} = [6, 7, -1]$ and $\vec{v} = [-1, 1, 1]$.
   (e) $\vec{u} = [3, -6, -1]$ and $\vec{v} = [-1, 2, -3]$.

5. Let $\vec{u} = [2, 2, -1]$. Find

   (a) the unit vector in the direction of $\vec{u}$,
   (b) a vector whose length is 4 and whose direction is opposite to the direction of $\vec{u}$.

6. Take two vectors $\vec{u}, \vec{v} \in \mathbf{R^n}$ such $\vec{u}, \vec{v} \neq \vec{0}$ and they are neither parallel nor perpendicular. An important thing is to decompose $\vec{u}$ into two components - parallel to $\vec{v}$ and perpendicular to $\vec{v}$. This means that

   $\vec{u} = \overrightarrow{u_{\vec{v}}} + \overrightarrow{u_{\perp\vec{v}}}$, where $\overrightarrow{u_{\vec{v}}} \parallel \vec{v}$ and $\overrightarrow{u_{\perp\vec{v}}} \perp \vec{v}$.

   $\vec{v}$ plays the role of a reference direction and $\overrightarrow{u_{\vec{v}}}$ is called the projection of $\vec{u}$ onto $\vec{v}$.

   (a) Show that $\overrightarrow{u_{\vec{v}}} = \frac{\vec{u} \circ \vec{v}}{|\vec{v}|^2} \cdot \vec{v}$ and, hence, $\overrightarrow{u_{\perp\vec{v}}} = \vec{u} - \frac{\vec{u} \circ \vec{v}}{|\vec{v}|^2} \cdot \vec{v}$.
   (b) Find this decomposition for

      i. $\vec{u} = [1, -1]$ if $\vec{v} = [2, 3]$,
      ii. $\vec{u} = [1, 2, 3]$ if $\vec{v} = [-1, 1, 2]$.

7. Find the exact value of the angle between $\vec{u}$ and $\vec{v}$.

   (a) $\vec{u} = [2, 1]$ and $\vec{v} = [-3, 1]$.
   (b) $\vec{u} = 2\vec{i} - 3\vec{j}$ and $\vec{v} = 5\vec{i} + 2\vec{j}$.
   (c) $\vec{u} = [1, 0, 1]$ and $\vec{v} = [1, 1, 0]$.
   (d) $\vec{u} = 2\vec{i} - \vec{j} + \vec{k}$ and $\vec{v} = -\vec{i} - 3\vec{k}$.

8. Let $A = (1, 2)$. Find all points B from the line $x = -1$ for which the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is equal to $45°$.

9. ($\star$) Let $\vec{u} = [1, 1, p]$ and $\vec{v} = [-1, p, 1]$, $p \in \mathbf{R}$. Find all possible angles between $\vec{u}$ and $\vec{v}$. Give the answer in a form $[\alpha, \beta]$, where α and β are given exactly, in degrees.

10. Let $\vec{u}, \vec{v} \in \mathbf{R^3}$ and let α be the angle between $\vec{u}$ and $\vec{v}$.

   (a) When $0 < \alpha < \pi$, applying the cosine rule to a suitable triangle, prove, formally, that $\vec{u} \circ \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\alpha$.

   (b) Complete the proof for $\alpha = 0$ and for $\alpha = \pi$.

11. Find all unit vectors that are perpendicular both to $\vec{u} = [1, 2, 1]$ and $\vec{v} = [1, 1, -1]$ using

   (a) scalar product,

   (b) vector product.

12. $ABCD$ is a parallelogram such that $A = (1, 0, 2)$, $B = (2, 3, 5)$, $C = (1, 2, 7)$ and $\overrightarrow{AB} = \overrightarrow{CD}$. Find the coordinates of D and the area of the parallelogram.

13. $\vec{u} = [1, 2, 1]$, $\vec{v} = [1, 1, -1]$ and $\vec{w} = [-1, 2, 5]$ start at the same point A and span a parallelepiped. Find its volume and the height between the walls that are spanned by $\vec{u}$ and $\vec{v}$

14. Let $A = (1, 0, 2)$, $B = (2, 3, 5)$ and $C = (1, 2, 7)$. Find all points D from the Z-axis for which

   (a) the volume of tetrahedron $ABCD$ is equal to 3,

   (b) A, B, C, D are co-planar.

15. Prove, formally, that $(\vec{u} \times \vec{v}) \circ \vec{w} = \det \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix}$ for $\vec{u}, \vec{v}, \vec{w} \in \mathbf{R^3}$.

16. ($\star$) Let $\vec{u}, \vec{v} \in \mathbf{R^3}$ be non-zero vectors.

   (a) Show that $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 \cdot |\vec{v}|^2 - (\vec{u} \circ \vec{v})^2$.

   (b) Hence prove that $|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \cdot \sin\alpha$, where α is the angle between $\vec{u}$ and $\vec{v}$.

17. Let $A, B, C \in \mathbf{R^2}$ be vertices of a triangle. Show that if each coordinate of these points is a rational number then the area of the triangle is also rational.

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