Mathematical logic and sets

1. Show that the following formulas are laws of logic.

   (a) $[\neg(p \vee q)] \Leftrightarrow [(\neg p) \wedge (\neg q)]$.
   (b) $[\neg(p \wedge q)] \Leftrightarrow [(\neg p) \vee (\neg q)]$.
   (c) $[\neg(p \Leftrightarrow q)] \Leftrightarrow [(\,(\neg p) \wedge q\,) \vee (\,(\neg q) \wedge p\,)]$.
   (d) $[(p \Rightarrow q) \wedge (q \Rightarrow r)] \Rightarrow (p \Rightarrow r)$.

2. Write the negation of the following statements in the way that the final answer does not contain any negation sign.

   (a) $x = 2 \vee x = 3$.
   (b) $x \leq 2 \vee x \geq 3$.
   (c) $x \geq 2 \wedge x < 3$.
   (d) $n \in \mathbf{N} \wedge n \geq 13 \wedge n < 20$.
   (e) $4 < x < 7 \vee x \leq 10$.
   (f) $k \in \mathbf{Z} \wedge (k \leq -6 \vee k \geq 25)$.
   (g) $x > 3 \Rightarrow x^2 > 9$.
   (h) $x^2 < 4 \Leftrightarrow -2 < x < 2$.
   (i) I'll eat cake but I'll not drink tea.
   (j) If f is an increasing function then $2f$ is also an increasing function.
   (k) I'll help my brother iff I have some free time.

3. Using quantifiers, operations of logic, symbols and numbers write the statements below.

   (a) $a_n = \dfrac{n+1}{n+2}$ is increasing.
   (b) $d_n = \dfrac{2^n}{n^2}$ is non-increasing from the 2nd element.
   (c) Equation $x^3 - 3x + 8 = 0$ has a real solution.
   (d) Equation $x^3 - 3x + 8 = 0$ has exactly one solution.
   (e) The system
   $$\begin{cases} x + 2y = -1 \\ x + y = 2 \\ x + 3y = 1 \end{cases}, \quad x, y \in \mathbf{R},$$
   has a solution.
   (f) The system
   $$\begin{cases} x + 2y = -1 \\ x + y = 2 \\ x + 3y = 1 \end{cases}, \quad x, y \in \mathbf{R},$$
   has no solution.
   (g) 14371 is a prime number.
   (h) $\mathbf{R} \setminus \{0\}$ is the set of values of $f(x) = \dfrac{1}{x-1}$, $x \neq 1$.
   (i) 2 is the smallest element of a sequence $a_n = n! + 1$.
   (j) The greatest common divisor of 8, 12 and 22 is 2.
   (k) The least common multiple of 3, 4 and 6 is 12.

4. Write the negation of the following theorems in the way that the final answer does not contain any negation sign.

(a) $\forall x \in \mathbf{R}\ \ x^2 + 2x < 0.$

(b) $\forall x \in \mathbf{Z}\ \ 2^x \neq 3.$

(c) $\exists x \in \mathbf{R}\ \ x^2 + 2x < 0.$

(d) $\exists x \in \mathbf{Z}\ \ 2^x \neq 3.$

(e) $\forall x \in D_f\ \ f(-x) = f(x)$
(this means that f is an even function over its domain D_f).

(f) $\forall x_1, x_2 > 0\ \ x_2 > x_1 \Rightarrow \frac{1}{\sqrt{x_2}} > \frac{1}{\sqrt{x_1}}$
(this means that $y = \frac{1}{\sqrt{x}}$ is decreasing in $(0, \infty)$).

(g) $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R}\ \ x^2 + y^3 = 0.$

(h) $\exists y \in \mathbf{R}\ \forall x \in \mathbf{R}\ \ x^2 + y^3 = 0.$

(i) $\forall y \in \mathbf{R}\ \exists x \in \mathbf{R}\ y = x^4 - x$
(this means that the set of values of $y = x^4 - x$ is $\mathbf{R}$).

(j) $\exists M \in \mathbf{R}\ \forall n \in \mathbf{N}^+\ \ a_n \leq M,$
(this means that M is an upper boundary of a sequence a_n.

(k) $\forall \epsilon > 0\ \exists n_0 \in \mathbf{N}^+\ \forall n \in \mathbf{N}^+, n \geq n_0\ \ |a_n - L| \leq \epsilon.$
(this means that a number L is a limit of a sequence a_n).

5. Disprove the following theorems by giving adequate counterexamples.

(a) $\forall n \in \mathbf{N}\ \ 2^n \geq n^2.$

(b) $\forall n \in \mathbf{N}^+\ 2^n + 3$ is a prime number.

(c) $\forall n \in \mathbf{N}\ \ n^6 - n$ is divisible by 6.

(d) $\forall x \in \mathbf{R}\ \ 9x^2 + 12x + 4 > 0.$

(e) $\forall x_1, x_2 \in \mathbf{R} \setminus \{0\}\ \ x_2 > x_1 \Rightarrow \frac{1}{x_2} < \frac{1}{x_1}.$

(f) $\forall x, y > 0\ \ xy > 1 \Rightarrow (x > 1 \wedge y > 1).$

(g) $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R}\ \ xy \neq 0.$

(h) If a_n is an arithmetic sequence then $|a_n|$ is also an arithmetic sequence.

(i) If $|a_n|$ is a geometric sequence then a_n is also a geometric sequence.

(j) If $a_n > 0$ for every $n \in \mathbf{N}^+$ and $\lim_{n\to\infty} a_n$ exists then $\lim_{n\to\infty} a_n > 0$.

(k) If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ then $\lim_{n\to\infty} (a_n - b_n) = 0$.

(l) If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n$ does not exist then $\lim_{n\to\infty} (a_n \cdot b_n)$ is 0 or it does not exist

6. Prove by contradiction the following theorems.

(a) $\sqrt{5}$ is irrational.

(b) $\log_2 3$ is irrational.

(c) If $x \in \mathbf{Q} \setminus \{0\}$ and $y \notin \mathbf{Q}$ then $xy \notin \mathbf{Q}$.

(d) $A \subset B\ \Rightarrow A \setminus B = \emptyset$.

(e) If $\lim_{n\to\infty} a_n = \infty$ or $-\infty$ and $\lim_{n\to\infty} b_n$ does not exist then $\lim_{n\to\infty} \frac{b_n}{a_n}$ is 0 or it does not exist.

7. Prove the following theorems by any method.

(a) Let $x, y > 0,\ y \neq 1$. Then
$\log_y x \in \mathbf{Q} \ \Leftrightarrow\ \exists a > 0, a \neq 1\ \exists p \in \mathbf{Q}\ \exists q \in \mathbf{Q} \setminus \{0\}\ \ x = a^p, y = a^q.$

(b) a_n is a arithmetic sequence iff
$\exists A, B \in \mathbf{R}\ \forall n \in \mathbf{N}^+\ \ a_n = An + B.$

(c) If a_n is a geometric sequence then $|a_n|$ is also a geometric sequence.

(d) Let f be an even or an odd function. Then

- if x_0 is its root then $-x_0$ is also its root,
- if it is monotonic in some interval (a, b), $a < b$, then it is monotonic in $(-b, -a)$,
- if it has a turning point at $x = x_0$ then it has a turning point at $x = -x_0$,
- if $x = x_0$ is its asymptote then $x = -x_0$ is also its asymptote.

8. Prove by induction that for all suitable natural n

(a) $1^3 + 2^3 + 3^3 + ... + n^3 = \dfrac{n^2(n+1)^2}{4}$,

(b) $1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n-1}n^2 = \dfrac{1}{2}(-1)^{n-1}n(n+1)$,

(c) $4^n + 2$ is divisible by 3,

(d) $n^7 - n$ is divisible by 7,

(e) $n! > n^2$ for $n \geq 4$,

(f) $\dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + ... + \dfrac{1}{n^2} \leq 2 - \dfrac{1}{n}$.

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