Theorem and proofs

1. Write the negation of the following statements in the way that the final answer does not contain any negation sign.

    1. $x = 2 \vee x = 3$.
    2. $x = 5 \vee x = 8 \vee x = 100$.
    3. $x \leq 2 \vee x \geq 3$.
    4. $x < 12 \vee x > 53$.
    5. $x \geq 2 \wedge x < 3$.
    6. $-5 < x \leq 0$.
    7. $n \in \mathbf{N} \wedge n \geq 13 \wedge n < 20$.
    8. $(2 + 2 \cdot 2 = 8) \wedge$ (5 is a prime number).
    9. $4 < x < 7 \vee x \leq 10$.
    10. $(x < -\sqrt{2} \vee x > 3\pi) \wedge x \geq -2$.
    11. $k \in \mathbf{Z} \wedge (k \leq -6 \vee k \geq 25)$.
    12. $x^2 > 9 \Rightarrow x > 3$.
    13. $x > 3 \Rightarrow x^2 > 9$.
    14. $x^2 < 4 \Leftrightarrow x < 2$.
    15. $x^2 < 4 \Leftrightarrow -2 < x < 2$.
    16. I'll buy a car or a motorcycle.
    17. I'll eat cake but I'll not drink tea.
    18. If f is an increasing function then $2f$ is also an increasing function.
    19. If tomorrow is sunny I'll go for a walk.
    20. f is an increasing function iff $2f$ is an increasing function.
    21. I help my brother iff I have some free time.

2. Using quantifiers, logical operations, symbols and numbers write the statements below.

    1. $a_n = \dfrac{n+1}{n+2}$ is increasing.
    2. $b_n = \dfrac{n!}{50^n}$ is decreasing.
    3. $c_n = \dfrac{7^n}{n!}$ is non-decreasing from the 6th element.
    4. $d_n = \dfrac{2^n}{n^2}$ is non-increasing from the 2nd element.
    5. Equation $x^3 - 3x + 8 = 0$ has a real solution.
    6. Equation $x^3 - 3x + 8 = 0$ has exactly one solution.
    7. Equation $x^4 + 4x + 8 = 0$ has no real solution.
    8. Equation $x^4 + 4x + 8 = 0$ has exactly one positive solution.
    9. The system
    $$\begin{cases} x + 2y = -1 \\ x + y = 2 \\ x + 3y = 1 \end{cases}, \quad x, y \in \mathbf{R},$$
    has a solution.

10. The system
$$\begin{cases} x+2y+z=-1 \\ x+y+3z=2 \end{cases}, \; x,y,z \in \mathbf{R},$$
has a solution.
11. The system
$$\begin{cases} x+2y=-1 \\ x+y=2 \\ x+3y=1 \end{cases}, \; x,y \in \mathbf{R},$$
has no solution.
12. The system
$$\begin{cases} x+2y+z=-1 \\ x+y+3z=2 \\ x+3y+8z=1 \end{cases}, \; x,y,z \in \mathbf{R},$$
has no solution.
13. 2017 is a prime number.
14. 14371 is a prime number.
15. $\mathbf{R} \setminus \{0\}$ is the set of values of $f(x) = \dfrac{1}{x-1}$, $x \neq 1$.
16. $[0, \infty)$ is the set of values of $f(x) = \sqrt{x}$, $x \geq 0$.
17. 2 is the smallest element of a sequence $a_n = n! + 1$.
18. $\sqrt[3]{3}$ is the greatest element of a sequence $b_n = \sqrt[n]{n}$.
19. The greatest common divisor of 8, 12 and 22 is 2.
20. The greatest common divisor of 8, 12 and 27 is 1.
21. The least common multiple of 8, 12 and 16 is 48.
22. The least common multiple of 3, 4 and 6 is 12.

3. Write the negation of the following theorems in the way that the final answer does not contain any negation sign.

1. $\forall x \in \mathbf{R} \;\; x^2 + 2x < 0$.
2. $\forall n \in \mathbf{N} \;\; n^2 + 2n \geq 0$.
3. $\forall x \in \mathbf{Z} \;\; 2^x \neq 3$.
4. $\forall x \in \mathbf{Q} \;\; 3^x \neq 4$.
5. $\exists x \in \mathbf{R} \;\; x^2 + 2x < 0$.
6. $\exists n \in \mathbf{N} \;\; n^2 + 2n \geq 0$.
7. $\exists x \in \mathbf{Z} \;\; 2^x \neq 3$.
8. $\exists x \in \mathbf{Q} \;\; 3^x \neq 4$.
9. $\forall x \in D_f \;\; f(-x) = f(x)$
(this means that f is an even function over its domain D_f).
10. $\forall x_1, x_2 \in \mathbf{R} \;\; x_2 > x_1 \Rightarrow x_2^3 > x_1^3$
(this means that $y = x^3$ is increasing in $\mathbf{R}$.
11. $\forall x_1, x_2 > 0 \;\; x_2 > x_1 \Rightarrow \dfrac{1}{\sqrt{x_2}} > \dfrac{1}{\sqrt{x_1}}$
(this means that $y = \dfrac{1}{\sqrt{x}}$ is decreasing in $(0, \infty)$).
12. $\forall x \in \mathbf{R} \; \exists y \in \mathbf{R} \;\; x^2 + y^3 = 0$.
13. $\exists y \in \mathbf{R} \; \forall x \in \mathbf{R} \;\; x^2 + y^3 = 0$.

14. $\forall x \in \mathbf{R}\ \exists n \in \mathbf{N} \quad nx = 0.$
15. $\exists n \in \mathbf{N}\ \forall x \in \mathbf{R} \quad nx = 0.$
16. $\forall y \in \mathbf{R}\ \exists x \in \mathbf{R}\ y = x^4 - x$
    (this means that the set of values of $y = x^4 - x$ is $\mathbf{R}$).
17. $\forall y \in [-1,1]\ \exists x \in \mathbf{R} \quad y = \sin x$
    (this means that the set of values of $y = \sin x$ is $[-1,1]$).
18. $\exists M \in \mathbf{R}\ \forall n \in \mathbf{N}^+ \quad a_n \leq M,$
    (this means that M is an upper boundary of a sequence a_n.
19. $\exists n \in \mathbf{R}\ \forall n \in \mathbf{N}^+ \quad b_n \geq m.$
    (this means that m is a lower boundary of a sequence b_n.
20. $\forall \epsilon > 0\ \exists n_0 \in \mathbf{N}^+\ \forall n \in \mathbf{N}^+, n \geq n_0 \quad |a_n - L| \leq \epsilon.$
    (this means that a number L is a limit of a sequence a_n).
21. $\forall r > 0\ \exists n_0 \in \mathbf{N}^+\ \forall n \in N^+, n \geq n_0 \quad a_n > r.$
    (this means that $\lim_{n\to\infty} a_n = \infty$).

4. Disprove the following theorems by giving adequate counterexamples.

   1. $\forall n \in \mathbf{N} \quad 2^n \geq n^2.$
   2. $\forall n \in \mathbf{N}, n \geq 3 \quad 3^n > n!.$
   3. $\forall n \in \mathbf{N}^+$ $2^n + 3$ is a prime number.
   4. $\forall n \in \mathbf{N}^+$ $n^2 + n + 1$ is a prime number.
   5. $\forall n \in \mathbf{N}$ $n^6 - n$ is divisible by 6.
   6. $\forall n \in \mathbf{N}$ $n^{15} - 15n$ is divisible by 15.
   7. $\forall x \in \mathbf{R} \quad 9x^2 + 12x + 4 > 0.$
   8. $\forall x \in \mathbf{R} \quad |6x^2 + 7x + 1| > 0.$
   9. $\forall x_1, x_2 \in \mathbf{R} \setminus \{0\} \quad x_2 > x_1 \Rightarrow \frac{1}{x_2} < \frac{1}{x_1}.$
   10. $\forall x_1, x_2 \in \mathbf{R} \quad x_2^4 = x_1^4 \Rightarrow x_2 = x_1.$
   11. $\forall x, y > 0 \quad xy > 1 \Rightarrow (x > 1 \wedge y > 1).$
   12. $\forall x, y, a, b \in \mathbf{R} \quad (x > a \wedge y > b) \Rightarrow xy > ab.$
   13. $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R} \quad xy \neq 0.$
   14. $\forall x \in \mathbf{R}\ \exists y \in \mathbf{R} \quad x + y^2 = 0.$
   15. If a_n is an arithmetic sequence then $|a_n|$ is also an arithmetic sequence.
   16. If $|a_n|$ is an arithmetic sequence then a_n is also an arithmetic sequence.
   17. If $|a_n|$ is a geometric sequence then a_n is also a geometric sequence.
   18. If $a_n > 0$ for every $n \in \mathbf{N}^+$ and $\lim_{n\to\infty} a_n$ exists then $\lim_{n\to\infty} a_n > 0$.
   19. If $b_n < 1$ for every $n \in \mathbf{N}^+$ and $\lim_{n\to\infty} b_n$ exists then $\lim_{n\to\infty} b_n < 1$.
   20. If a_n has no upper boundary then $\lim_{n\to\infty} a_n = \infty$.
   21. If a_n has no lower boundary then $\lim_{n\to\infty} a_n = -\infty$.
   22. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ then $\lim_{n\to\infty} (a_n - b_n) = 0$.
   23. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$ then $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$.
   24. If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = \infty$ then $\lim_{n\to\infty} (a_n \cdot b_n)$ is 0 or ∞.
   25. If $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n$ does not exist then $\lim_{n\to\infty} (a_n \cdot b_n)$ is 0 or it does not exist
       (compare it to a similar theorem in task 5).
   26. $\forall x, y \in \mathbf{R} \quad \lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$, where $\lfloor\ \rfloor$ denotes the 'floor' function.

5. Prove by contradiction the following theorems.

   1. $\sqrt{5}$ is irrational.
   2. $\sqrt[3]{2}$ is irrational.
   3. $\sqrt[5]{40}$ is irrational.
   4. $\log_2 3$ is irrational.
   5. $\log_{50} 40$ is irrational.
   6. $\log_{0.7} 5$ is irrational.
   7. If $x \in \mathbf{Q}$ and $y \notin \mathbf{Q}$ then $x - y \notin \mathbf{Q}$.
   8. If $x \in \mathbf{Q} \setminus \{0\}$ and $y \notin \mathbf{Q}$ then $xy \notin \mathbf{Q}$.
   9. $A \subset B \;\Rightarrow A \setminus B = \emptyset$.
   10. $A \setminus B = \emptyset \;\Rightarrow A \subset B$. Therefore, $A \subset B \;\Leftrightarrow A \setminus B = \emptyset$.
   11. $A \subset B \;\Leftrightarrow A \cap B = A$.
   12. If $\lim\limits_{n\to\infty} a_n = \infty$ or $-\infty$ and $\lim\limits_{n\to\infty} b_n$ does not exist then $\lim\limits_{n\to\infty} \frac{b_n}{a_n}$ is 0 or it does not exist.
   13. If $\lim\limits_{n\to\infty} a_n$ is 0^+ or 0^- and $\lim\limits_{n\to\infty} b_n$ does not exist then $\lim\limits_{n\to\infty} (a_n \cdot b_n)$ is 0 or it does not exist (compare it to a similar theorem in task 4).

6. Prove the following theorems by any method.

   1. Let $x, y > 0,\ y \neq 1$. Then
      $\log_y x \in \mathbf{Q} \;\Leftrightarrow\; \exists a > 0, a \neq 1\ \exists p \in \mathbf{Q}\ \exists q \in \mathbf{Q} \setminus \{0\}\ \ x = a^p, y = a^q$.
   2. a_n is a arithmetic sequence iff
      $\exists A, B \in \mathbf{R}\ \forall n \in \mathbf{N}^+\ \ a_n = An + B$.
   3. a_n is a geometric sequence iff
      $(\forall n \in \mathbf{N}^+, n \geq 2\ \ a_n = 0) \vee (\exists A \in \mathbf{R}\ \exists r \in \mathbf{R} \setminus \{0\}\ \forall n \in \mathbf{N}^+\ \ a_n = A \cdot r^n)$.
   4. $S_n = a_1 + a_2 + ... + a_n$ is an arithmetic series iff
      $\exists A, B \in \mathbf{R}\ \forall n \in \mathbf{N}^+\ \ S_n = An^2 + Bn$.
   5. $S_n = a_1 + a_2 + ... + a_n$ is a geometric series iff
      $(\ (\forall n \in \mathbf{N}^+\ \ S_n = na_1) \vee (\exists A \in \mathbf{R}\ \exists r \in \mathbf{R} \setminus \{1\}\ \forall n \in \mathbf{N}^+\ \ S_n = A - A \cdot r^n)\ )$.
   6. If a_n is a geometric sequence then $|a_n|$ is also a geometric sequence.
   7. Let $a > 0, a \neq 1$. Then if a_n is an arithmetic sequence then a^{a_n} is a geometric sequence.
   8. Let $a > 0, a \neq 1$. Then if a_n is a positive geometric sequence then $\log_a a_n$ is an arithmetic sequence.
   9. For any determinant of dimension 3 the Sarrus rule is equivalent to the so-called Laplace expansion with respect to the first row, that is,
      $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$
   10. Consider a system of two linear equations with two variables x and y, written in the matrix form $A \cdot X = B$. Then for this special case
       $\det(A) = \det(A_x) = \det(A_y) = 0 \Leftrightarrow$ the system has infinitely many solutions
       (compare it to the general case of n equations with n variables).
   11. $\forall x \neq 0\ \ \operatorname{sgn}(x) = \frac{x}{|x|} = \frac{|x|}{x}$.
   12. $\forall x \in \mathbf{R}\ \forall n \in \mathbf{Z}\ \ \lfloor x + n \rfloor = \lfloor x \rfloor + n$, where $\lfloor\ \rfloor$ denotes the 'floor' function.

13. Let f be an even or an odd function. Then
    - if x_0 is its root then $-x_0$ is also its root,
    - if it is monotonic in some interval (a, b), $a < b$, then it is monotonic in $(-b, -a)$,
    - if it has a turning point at $x = x_0$ then it has a turning point at $x = -x_0$,
    - if $x = x_0$ is its asymptote then $x = -x_0$ is also its asymptote.
14. If f is an odd function and $f(0)$ exists then $f(0) = 0$.
15. The only function that is both even and odd is constantly equal to 0.
16. Let f, g be two functions. Then
    - if f and g are even then $f + g$ and $f - g$ are even,
    - if f and g are odd then $f + g$ and $f - g$ are odd,
    - if f and g are even or odd then $f \cdot g$, $\frac{f}{g}$ and $\frac{g}{f}$ are even,
    - if f is even and g is odd then $f \cdot g$, $\frac{f}{g}$ and $\frac{g}{f}$ are odd,
17. Consider a composite function $h = f \circ g$. Then
    - if g is even then h is even,
    - if g is odd and f is even then h is even,
    - if g is odd and f is odd then h is odd,
    - if g is periodic then h is periodic.
18. Every function f which satisfies the condition $(x \in D_f \Rightarrow -x \in D_f)$ can be written uniquely as a sum of two functions of which one is even and the second one is odd.
19. If f is an even function and $y = ax + b$ is its asymptote at $+\infty$ (or $-\infty$) then $y = -ax + b$ is its asymptote at at $-\infty$ (or $+\infty$).
20. If f is an odd function and $y = ax + b$ is its asymptote at $+\infty$ (or $-\infty$) then $y = -ax - b$ is its asymptote at at $-\infty$ (or $+\infty$).
21. If for some $a,\ b \in \mathbf{R}$, $\lim_{x \to \infty} (f(x) - ax) = b$ then $y = ax + b$ is an asymptote of f at ∞.

    Analogously at $-\infty$.
22. If $y = ax + b$ is an asymptote of f then $y = (A + a)x + (B + b)$ is an asymptote of g, where $g(x) = f(x) + Ax + B$.

Tasks 4, 5 and 6 are going to be developed throught the whole course.