

# MULTIPLE SOLUTIONS FOR DIRICHLET NONLINEAR BVPS INVOLVING FRACTIONAL LAPLACIAN

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ABSTRACT. The existence of at least two solutions to superlinear integral equation in cone is proved using the Krasnosiel'skii Fixed Point Theorem. The result is applied to the Dirichlet BVPs with the fractional Laplacian.

## 1. INTRODUCTION AND MOTIVATION

It is well known that the superlinear equation with  $p > 1$  on the real line

$$u = bu^p + u_0 \quad (1)$$

can have none, one or more solutions  $u$  depending on the data  $b > 0$  and  $u_0 \geq 0$ . For example, if we additionally assume that

$$bu_0^{p-1} < c_p \quad (2)$$

for some constant

$$c_p = \left( (p-1)^{\frac{1-p}{p}} + (p-1)^{\frac{1}{p}} \right)^{-p} \quad (3)$$

then the existence of at least two nonnegative solutions of (1) is guaranteed, since thus the minimum of the function  $bu^{p-1} + u_0u^{-1}$  is ascertained to be smaller than the constant 1.

In this paper we would like to show that this simple observation can be generalized if we replace power term  $bu^p$  defined on the real line with a power like nonlinearity in a Banach space under some additional, suitable conditions like coercivity and compactness on some cone in this Banach space. More specifically, we shall consider the equation in the cone  $P$  in the Banach space  $E$  with the norm  $|\cdot|$  in the form

$$u = B(u) + u_0 \quad (4)$$

for some given element  $u_0 \in P$  and  $p$ -power, coercive and compact form  $B$  defined on  $P$ . The assumption (2) guaranteeing the existence of at least two solutions for the quadratic equation (1) now has to be adequately rephrased for (4) as

$$b|u_0|^{p-1} < c_p \quad (5)$$

where  $b > 0$  denotes the best estimate such that for any  $u \in P$

$$|B(u)| \leq b|u|^p. \quad (6)$$

Our main theoretical tool for the application to the superlinear integral equations and the BVPs with the fractional Laplacian is the following theorem.

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**Theorem 1.1.** *Assume that, for any given cone  $P \subset E$ , a compact mapping  $B : P \rightarrow P$  satisfies the following condition*

$$a|u|^p \leq |B(u)| \leq b|u|^p, \quad (7)$$

*for some  $b > a > 0$ . Then for any  $u_0 \in P$  as small as to satisfy (5) the equation (4) admits at least two solutions in  $P$ .*

As a direct but nontrivial application of this result we shall obtain among other applications a multiplicity result for the following superlinear boundary value problem involving the one-dimensional fractional Laplacian.

$$(-\Delta)^{\alpha/2}u(x) = (u(x))^p + h(x), \quad \text{for } x \in (-1, 1), \quad (8)$$

$$u(x) = 0, \quad \text{for } |x| \geq 1. \quad (9)$$

We shall denote by  $G_{(-1,1)}$  both the Green function and the Green operator corresponding to the Dirichlet linear problem on  $(-1, 1)$  for the fractional Laplacian (see Preliminaries). We say that  $u : [-1, 1] \rightarrow [0, \infty)$  is symmetric and unimodal on  $[-1, 1]$  iff  $u(x) = u(-x)$  for all  $x \in [-1, 1]$ ,  $u$  is nondecreasing on  $[-1, 0]$  and nonincreasing on  $[0, 1]$ .  $BC([-1, 1])$  denotes the space of all bounded continuous functions  $f : [-1, 1] \rightarrow \mathbb{R}$  with the standard supremum norm over the interval  $[-1, 1]$ .

**Theorem 1.2.** *Let  $\alpha \in (1, 2)$ ,  $p > 1$  and  $h \in BC([-1, 1])$  be a nonnegative, symmetric and unimodal function on  $[-1, 1]$ . Assume also that (5) is satisfied where  $u_0 = G_{(-1,1)}h$  and  $Bu = G_{(-1,1)}u^p$ . Then there exist at least two nonnegative weak solutions to the boundary value problem (8-9). Moreover, if  $h$  is regular enough, i.e.  $h \in C^\gamma(-1, 1)$  with  $\gamma > 2 - \alpha$  then the solutions are classical.*

The proofs of the above theorems will be postponed to the next sections.

The motivation for the fractional Laplacian originates from multiple sources, among others from: Probability and Mathematical Finance as the infinitesimal generators of stable Lévy processes ([4, 5, 8]), which play nowadays an important role in stochastic modeling in applied sciences and in financial mathematics, Mechanics encountered in elastostatics as Signorini obstacle problem in linear elasticity ([12]) and finally from Fluid Mechanics as quasi-geostrophic fractional Navier-Stokes equation, see [13, 35] and references therein and Phase Transitions as described in [27]. Let us also mention here that the result corresponding to Theorem 1.1 for the equations involving bilinear form, corresponding to  $p = 2$ , were proved by one of the authors of this paper in [33] motivated by the Navier–Stokes equation (cf. [14]), the Boltzmann equation (cf. [23]), the quadratic reaction diffusion equation (cf. [18]), the Smoluchowski coagulation equation (cf. [28]) or the system modeling chemotaxis [32] to name but a few. The problem of uniqueness of solutions for these equations attracted a lot of attention and only some partial results are known. In some cases nonuniqueness occurs and the existence of two solutions can be proved. Sometimes one of the solution is a trivial one and then the proof relies on finding a nontrivial one, which can be of lower regularity or a nonstable one. In these models one encounters another problem making our approach not feasible i.e. very common lack of compactness, thus if we would like to make our approach feasible we are forced to consider some truncated baby model compatible with compact setting.

To prove the existence of two solutions we shall use the Krasnoselskii Fixed Point Theorem, cf. [16], which allows us to obtain more solutions if the nonlinear operator

has the required property of “crossing” identity twice, i.e. by the cone compression and the expansion on some appropriate subsets of the cone.

It should be noted that the problem of existence of multiple solutions of nonlinear equations was addressed by H. Amann in [1] in ordered Banach spaces rather than analysed from topological point of view as in our approach.

## 2. PRELIMINARIES CONCERNING FRACTIONAL LAPLACIAN

Let  $\alpha \in (0, 2)$  and  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  be a measurable function satisfying

$$\int_{\mathbb{R}^d} \frac{|u(x)|}{(1 + |x|)^{d+\alpha}} dx < \infty. \quad (10)$$

For such a function the fractional Laplacian can be defined as follows (cf. [6], page 61)

$$(-\Delta)^{\alpha/2} u(x) = c_{d,-\alpha} \lim_{\varepsilon \rightarrow 0^+} \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} \frac{u(x) - u(y)}{|x-y|^{d+\alpha}} dy,$$

whenever the limit exists. Here we have

$$c_{d,\gamma} = \Gamma((d - \gamma)/2) / (2^\gamma \pi^{d/2} |\Gamma(\gamma/2)|).$$

It is known that if  $u$  satisfies (10) and  $u \in C^2(D)$  for some open set  $D \subset \mathbb{R}^d$  then  $(-\Delta)^{\alpha/2} u(x)$  is well defined for any  $x \in D$ , which can be justified by Taylor expansion of the function  $u$ . The fractional Laplacian may also be defined in a weak sense, see e.g. page 63 in [6].

Let us consider the Dirichlet linear problem for the fractional Laplacian on a bounded open set  $D \subset \mathbb{R}^d$

$$(-\Delta)^{\alpha/2} u(x) = g(x), \quad x \in D, \quad (11)$$

$$u(x) = 0, \quad x \notin D. \quad (12)$$

It is well known that there exist the Green operator  $G_D$  and the Green function  $G_D(x, y)$  corresponding to the problem (11)-(12). Namely, if  $g \in L^\infty$  then the unique (weak) solution of this problem is given by

$$u(x) = G_D g(y) = \int_D G_D(x, y) g(y) dy. \quad (13)$$

It should be noted that this  $u$  is in fact in  $C^\gamma$  with  $\gamma > 0$ , cf. [25], whence also follows that  $G_D$  increases interior regularity by  $\alpha$  on the level of the Hölder continuous functions. The definition and basic properties of the Green operator and the Green function may be found e.g. in [6] or [7]. It is well known that for any  $\alpha \in (0, 2)$  the Green function for the ball  $B(0, 1)$  is given by an explicit formula [5]

$$G_{B(0,1)}(x, y) = c_\alpha^d |x - y|^{\alpha-d} \int_0^{w(x,y)} r^{\alpha/2-1} (r+1)^{-d/2} dr, \quad x, y \in B(0, 1),$$

where

$$w(x, y) = (1 - |x|^2)(1 - |y|^2)|x - y|^{-2}$$

and

$$c_\alpha^d = \Gamma(d/2) / (2^\alpha \pi^{d/2} \Gamma^2(\alpha/2)).$$

We have  $G_{B(0,1)}(x, y) = 0$  if  $x \notin B(0, 1)$  or  $y \notin B(0, 1)$ .

In [25] some Krylov type estimates on the regularity of solutions to the equations involving fractional Laplacian were provided by X. Ros-Oton and J. Serra. The

regularity and the existence and uniqueness issues for the problems involving fractional Laplacian were also addressed by X. Cabré and Y. Sire [10, 11]. For any open bounded  $C^{1,1}$  domain  $D$ ,  $g \in L^\infty$  and a distance function  $\delta(x) = \text{dist}(x, \partial D)$  if  $u$  is the solution of the Dirichlet problem (11)-(12) then  $u/\delta^{\alpha/2}|_D$  can be continuously extended to  $\overline{D}$ . Moreover, we have  $u/\delta^{\alpha/2} \in C^\gamma(\overline{D})$  and we control the norm

$$\|u/\delta^{\alpha/2}\|_{C^\gamma(\overline{D})} \leq C|g|_\infty$$

for some  $\gamma < \min\{\alpha/2, 1-\alpha/2\}$ . It suffices, due to the compact embedding  $C^\gamma(\overline{D}) \subset C(\overline{D})$ , for compactness of the operator

$$G_D : C(\overline{D}) \rightarrow C(\overline{D}).$$

We say that the bounded measurable function  $u : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $\alpha$ -harmonic in an open set  $D \subset \mathbb{R}^d$  if  $(-\Delta)^{\alpha/2}u(x) = 0$ , for any  $x \in D$  (in the classical sense). It is known (see e.g. [6], [7]) that such a function  $u$  satisfies

$$u(x) = \int_{D^c} P_D(x, y)u(y) dy, \quad x \in D,$$

where  $P_D : D \times D^c \rightarrow \mathbb{R}$  is the Poisson kernel (corresponding to the fractional Laplacian). The Poisson kernel for a ball  $B(0, r) \subset \mathbb{R}^d$ ,  $r > 0$  is given by an explicit formula ([5])

$$P_{B(0,r)}(x, y) = C_\alpha^d \frac{(r^2 - |x|^2)^{\alpha/2}}{(|y|^2 - r^2)^{\alpha/2}|x - y|^d}, \quad |x| < r, |y| > r,$$

where  $C_\alpha^d = \Gamma(d/2)\pi^{-d/2-1} \sin(\pi\alpha/2)$ .

### 3. THE ABSTRACT MULTIPLICITY RESULT FOR COMPACT P-POWER OPERATORS

To prove Theorem 1.1 we shall follow the lines of the proof presented in [33] for  $p = 2$  and use the following theorem [16, Theorem 2.3.4] originating from the works of Krasnoselskii, cf. [20].

**Theorem 3.1.** *Let  $E$  be a Banach space, and let  $P \subset E$  be a cone in  $E$ . Let  $\Omega_1$  and  $\Omega_2$  be two bounded, open sets in  $E$  such that  $0 \in \Omega_1$  and  $\overline{\Omega_1} \subset \Omega_2$ . Let completely continuous operator  $T : P \rightarrow P$  satisfy conditions*

$$|Tu| \leq |u| \text{ for any } u \in P \cap \partial\Omega_1 \text{ and } |Tu| \geq |u| \text{ for any } u \in P \cap \partial\Omega_2$$

or, alternatively, the following two conditions

$$|Tu| \geq |u| \text{ for any } u \in P \cap \partial\Omega_1 \text{ and } |Tu| \leq |u| \text{ for any } u \in P \cap \partial\Omega_2$$

are satisfied. Then  $T$  has at least one fixed point in  $P \cap (\overline{\Omega_2} \setminus \Omega_1)$ .

*proof of Theorem 1.1.* Let us define the operator

$$Tu = B(u) + u_0 \tag{14}$$

then we shall apply Krasnosielkii Theorem once as a cone-compression in the neighborhood of zero and secondly as a cone-expansion at infinity.

Notice that we have the following estimates

$$\begin{aligned} |Tu| &\leq |u_0| + b|u|^p, \\ |Tu| &\geq |u_0| - b|u|^p, \end{aligned} \tag{15}$$

where constant  $b = |B| > 0$  denotes the the smallest constant  $b$  satisfying, for any  $u \in P$ , the inequality

$$|B(u)| \leq b|u|^p.$$

Then we can assume by (5) that there exists some intermediate value  $\rho_2 > 0$  such that

$$|u_0| + b\rho_2^p < \rho_2. \quad (16)$$

Indeed as announced in the introduction for the real line superlinear problem the above equation is equivalent to

$$|u_0|\rho_2^{-1} + b\rho_2^{p-1} < 1. \quad (17)$$

while the minimum of the function  $|u_0|\rho_2^{-1} + b\rho_2^{p-1}$  is attained at  $\rho_2$  such that  $\rho_2^p = |u_0|/(b(p-1))$  and the minimum value is equal to

$$|u_0|^{1-1/p} ((b(p-1))^{1/p} + b^{1/p}(p-1)^{(1-p)/p}). \quad (18)$$

Requiring the value (18) to be smaller than one as in (17) is equivalent to (5) which thus implies the claim (16).

Hence by (7) together with (16) for any  $u \in P$  and  $|u| = \rho_2$  one has

$$|Tu| \leq |u_0| + b|u|^p < \rho_2 = |u|. \quad (19)$$

Moreover, if  $u_0 = 0$  then  $u = 0$  is a solution. Otherwise, if  $u_0 \neq 0$  then for sufficiently small  $\rho_1$  such that  $\rho_2 > \rho_1 > 0$  and  $b\rho_1^p + \rho_1 < |u_0|$  for any  $u \in P$  and  $|u| = \rho_1$  one has

$$|Tu| \geq |u_0| - b\rho_1^p > \rho_1 = |u|. \quad (20)$$

Thus both conditions (19) and (20) can be accomplished if we assume  $b\rho_1^p + \rho_1 < |u_0| < \rho_2 - b\rho_2^p$ .

Finally, for sufficiently large values of  $\rho_3 > 0$  and any  $u \in P$  and  $|u| = \rho_3$ , due to the coercivity assumption (7)

$$|B(u)| \geq a|u|^p, \quad (21)$$

one obtains

$$|Tu| \geq a\rho_3^p - |u_0| > \rho_3 = |u|. \quad (22)$$

To be more specific  $\rho_3$  has to be so large that  $\rho_3 > \rho_2$  and  $|u_0| < a\rho_3^p - \rho_3$ .

Combining (19) with (20) we get that the intersection of the cone  $P$  with the spheres of the radii  $\rho_1$  and  $\rho_2$  (in the  $|\cdot|$  norm) is compressed while the one at the radii  $\rho_2$  and  $\rho_3$  is expanded yielding the desired two fixed points in each set. Note that it might be necessary to distinguish between  $\rho_2$  used in both sets as to prevent both fixed points to coincide. □

**Remark 1.** To guarantee (22) in fact it suffices to assume only that

$$\frac{|B(u)|}{|u|} \rightarrow \infty \text{ as } |u| \rightarrow \infty$$

instead of the lower estimate for the  $B(u)$  as in (7).

#### 4. MULTIPLICITY RESULT FOR SUPERLINEAR INTEGRAL OPERATOR INVOLVING P-POWER NONLINEARITY

Consider, for some open nonempty domain  $V \subset \mathbb{R}^d$ , the following equation in the space  $BC(\overline{V})$  of bounded and continuous functions defined as

$$GFu + u_0 = u \quad (23)$$

where  $u_0 \in BC(\overline{V})$  is given,  $u \in BC(\overline{V})$  is the unknown and  $G$  is some linear integral operator defined by

$$Gf(x) = \int_V G(x, y)f(y) dy \quad (24)$$

for some given kernel function  $G : \overline{V} \times \overline{V} \rightarrow \mathbb{R}$  smooth enough to guarantee compactness of  $G$  in  $BC(\overline{V})$ , while a nonlinear operator  $F$  is defined for  $p > 1$  by

$$Fu(y) = (u(y))^p. \quad (25)$$

Then the operator  $B$  from Theorem 1.1 can be defined as

$$B = GF. \quad (26)$$

Let us define for some given, nonempty and open set  $U \subset\subset V$  (i.e.  $U$  is such that  $\overline{U} \subset V$ ) and some constant  $\gamma_V > 0$  the cone  $P$  as

$$P = \{u \in BC(\overline{V}) : u \geq 0, \inf_U u \geq \gamma_V \sup_{\overline{V}} u\}. \quad (27)$$

Assume that the kernel  $G$  is positive on  $V \times V$  and that for any  $y \in \overline{V}$  the following property holds

$$\inf_{x \in U} G(x, y) \geq \gamma_V \sup_{x \in \overline{V}} G(x, y) \quad (28)$$

where  $\gamma_V > 0$  is independent of  $y$ .

Then the cone  $P$  is invariant under  $GF$ . Using standard arguments (see [33]) when we apply Theorem 1.1 for  $B = GF$  we arrive at the following theorem.

**Theorem 4.1.** *There exist at least two nonnegative solutions to the Hammerstein equation (23) provided the function  $G$  is regular enough to guarantee the compactness of the corresponding operator and satisfies (28), while  $F$  is defined by (25) for some  $p > 1$  and  $u_0$  is small enough as to satisfy (5).*

Note that to guarantee compactness of  $GF$  usually the domain  $U$  is assumed to be bounded and the kernel  $G$  smooth enough but also for unbounded  $U$  some results on compactness of  $G$  under stronger decay assumptions on  $F$  than the pure power like form were established, e.g. in [29].

#### 5. MULTIPLICITY RESULT FOR FRACTIONAL LAPLACIAN

In this section we prove Theorem 1.2. First we need two auxiliary lemmas. Let us denote  $V = (-1, 1)$ . Recall that  $G_V$  is the Green function for the one-dimensional problem (8)-(9), also denoting the corresponding Green operator.

**Lemma 5.1.** *Let  $a \in (0, 1)$ ,  $U = (-a, a)$ . There exists  $\gamma_U > 0$  such that for any  $y \in V$  we have*

$$\inf_{x \in U} G_V(x, y) \geq \gamma_U G_V(0, y).$$

*Proof.* For any  $x, y \in V$  by [7, Corollary 3.2] we have

$$c_\alpha \left( \frac{\delta^{\alpha/2}(x)\delta^{\alpha/2}(y)}{|x-y|} \wedge \delta^{\frac{\alpha-1}{2}}(x)\delta^{\frac{\alpha-1}{2}}(y) \right) \quad (29)$$

$$\leq G_V(x, y) \leq C_\alpha \left( \frac{\delta^{\alpha/2}(x)\delta^{\alpha/2}(y)}{|x-y|} \wedge \delta^{\frac{\alpha-1}{2}}(x)\delta^{\frac{\alpha-1}{2}}(y) \right), \quad (30)$$

where  $\delta(x) = \text{dist}(x, \partial V)$ ,  $a \wedge b = \min(a, b)$ .

Let  $x \in U$ ,  $y \in V$  be arbitrary. By (29) we get

$$G_V(x, y) \geq c_U(\delta^{\alpha/2}(y) \wedge \delta^{\frac{\alpha-1}{2}}(y)) = c_U\delta^{\alpha/2}(y).$$

On the other hand by (30) for any  $y \in V$  we have

$$\begin{aligned} G_V(0, y) &\leq C_\alpha \left( \frac{\delta^{\alpha/2}(0)\delta^{\alpha/2}(y)}{|y|} \wedge \delta^{\frac{\alpha-1}{2}}(0)\delta^{\frac{\alpha-1}{2}}(y) \right) \\ &= C_\alpha(\delta^{\alpha/2}(y)|y|^{-1} \wedge \delta^{\frac{\alpha-1}{2}}(y)). \end{aligned}$$

Hence for  $y \in (-1/2, 1/2)$  we get

$$G_V(0, y) \leq C_\alpha\delta^{\frac{\alpha-1}{2}}(y) \leq C_\alpha \leq 2^{\alpha/2}C_\alpha\delta^{\alpha/2}(y).$$

For  $y \in (-1, 1) \setminus (-1/2, 1/2)$  we obtain

$$G_V(0, y) \leq C_\alpha\delta^{\alpha/2}(y)|y|^{-1} \leq 2C_\alpha\delta^{\alpha/2}(y).$$

□

**Lemma 5.2.** *Suppose that  $f \in BC(\bar{V})$  is nonnegative, symmetric and unimodal on  $\bar{V}$ . Then  $G_V f \in BC(\bar{V})$  is also symmetric and unimodal on  $\bar{V}$ .*

*Proof.* Symmetry of  $G_V f$  follows by an explicit formula for the Green function of an interval (see Preliminaries). Note also that  $G_V f(-1) = G_V f(1) = 0$ . It is well known (see e.g. [25]) that  $G_V f$  is continuous on  $\bar{V}$ . Now we show that  $G_V f$  is nonincreasing on  $(0, 1)$ . To this end take any  $0 < x < y < 1$  and fix  $z = \frac{x+y}{2}$  and set  $r = 1 - z$ . Define the interval  $W = (z - r, z + r) = (z - r, 1)$ . By [6, p. 87] and [7, p. 318], for any  $w \in W$  we have

$$G_V f(w) = G_W f(w) + \int_{V \setminus W} G_V f(v) P_W(w, v) dv$$

where  $G_V, G_W$  are Green operators for  $V, W$  (respectively), while  $P_W$  is the Poisson kernel for  $W$ , all corresponding to the fractional Laplacian  $(-\Delta)^{\alpha/2}$  (see Preliminaries). Let  $\hat{w} = 2z - w$  be the inversion of a point  $w$  in respect to a point  $z$ . Clearly, we have  $\hat{x} = y$  and  $\hat{y} = x$ . Let us observe that

$$\int_{V \setminus W} G_V f(v) P_W(y, v) dv \leq \int_{V \setminus W} G_V f(v) P_W(x, v) dv.$$

Indeed, it follows from the fact, that for any  $v \in V \setminus W$  one has

$$\begin{aligned} P_W(y, v) &= \frac{C_\alpha^1 (r^2 - |y - z|^2)^{\alpha/2}}{|v - y| (|v - z|^2 - r^2)^{\alpha/2}} \\ &= \frac{C_\alpha^1 (r^2 - |x - z|^2)^{\alpha/2}}{|v - y| (|v - z|^2 - r^2)^{\alpha/2}} \\ &\leq \frac{C_\alpha^1 (r^2 - |x - z|^2)^{\alpha/2}}{|v - x| (|v - z|^2 - r^2)^{\alpha/2}} = P_W(x, v). \end{aligned}$$

Next we shall show that

$$G_W f(y) \leq G_W f(x).$$

Note that the Green function  $G_W$  satisfies the following symmetry properties for any  $v \in W$

$$G_W(\hat{y}, \hat{v}) = G_W(y, v), \quad (31)$$

$$G_W(\hat{y}, v) = G_W(y, \hat{v}). \quad (32)$$

Put  $W_+ = (z, 1)$  and  $W_- = (2z - 1, z)$ . It follows that

$$\begin{aligned} G_W f(y) &= \int_W G_W(y, v) f(v) dv \\ &= \int_{W_+} G_W(y, v) f(v) dv + \int_{W_-} G_W(y, v) f(v) dv \\ &= \int_{W_+} G_W(y, v) f(v) dv + \int_{W_+} G_W(y, \hat{v}) f(\hat{v}) dv. \end{aligned}$$

Similarly using  $x = \hat{y}$  one obtains

$$\begin{aligned} G_W f(x) &= \int_W G_W(\hat{y}, v) f(v) dv \\ &= \int_{W_+} G_W(\hat{y}, v) f(v) dv + \int_{W_-} G_W(\hat{y}, v) f(v) dv \\ &= \int_{W_+} G_W(\hat{y}, v) f(v) dv + \int_{W_+} G_W(\hat{y}, \hat{v}) f(\hat{v}) dv. \end{aligned}$$

Using the above relation and again (31)-(32) we get

$$\begin{aligned} &G_W f(y) - G_W f(x) \\ &= \int_{W_+} (G_W(y, v) - G_W(\hat{y}, v)) f(v) dv + \int_{W_+} (G_W(\hat{y}, v) - G_W(y, v)) f(\hat{v}) dv \\ &= \int_{W_+} (G_W(y, v) - G_W(\hat{y}, v)) (f(v) - f(\hat{v})) dv \leq 0, \end{aligned}$$

since for any  $v \in W_+$

$$\begin{aligned} f(v) - f(\hat{v}) &\leq 0, \\ G_W(y, v) - G_W(\hat{y}, v) &\geq 0, \end{aligned}$$

by Corollary 3.2 from [19]. It follows that  $G_V f$  is nonincreasing on  $(0, 1)$ .  $\square$



*proof of Theorem 1.2.* The problem can be formulated as required

$$u = G_V F u + u_0 \quad (33)$$

where

$$u_0(x) = G_V h(x) \quad (34)$$

and

$$G_V f(x) = \int_V G_V(x, y) f(y) dy, \quad F u(x) = u(x)^p, \quad (35)$$

where  $G_V(x, y)$  is the Green function for  $V$ .

Let  $a \in (0, 1)$ ,  $U = (-a, a)$  and  $\gamma_U$  be the constant from Lemma 5.1. Let us define for the given  $a$  the cone  $P$  in the space of bounded and continuous functions  $BC(\overline{V})$ :

$$P = \{u \in BC(\overline{V}) : u \geq 0, \inf_U u \geq \gamma_U \sup_{\overline{V}} u, u \text{ is symmetric and unimodal on } \overline{V}\}.$$

We will show that the cone  $P$  is invariant under  $B = G_V F$ . Indeed,  $B$  maps the set of bounded, continuous and nonnegative functions on  $\overline{V}$  into itself. Lemma 5.2 gives that  $B$  preserves symmetry and unimodality. What is more, for any  $x \in U$  by Lemma 5.1 we have

$$\begin{aligned} B(u)(x) &= \int_{-1}^1 G_V(x, y) u^p(y) dy \\ &\geq \gamma_U \int_{-1}^1 G_V(0, y) u^p(y) dy = \gamma_U B(u)(0). \end{aligned}$$

It follows that  $P$  is invariant under  $B = GF$ .

$B = G_V F$  satisfies the following coercivity condition with sup norms

$$\begin{aligned} \inf_{|u|=1, u \in P} |B(u)| &= \inf_{|u|=1, u \in P} \sup_{x \in V} \int_{-1}^1 G_V(x, y) u^p(y) dy \\ &\geq \inf_{|u|=1, u \in P} \int_{-a}^a G_V(0, y) u^p(y) dy \\ &\geq \inf_{|u|=1, u \in P} \int_{-a}^a G_V(0, y) \gamma_U^p |u|^p dy \\ &\geq \gamma_U^p \int_{-a}^a G_V(0, y) dy > 0. \end{aligned}$$

We also have

$$\begin{aligned} \sup_{|u|=1, u \in P} |B(u)| &= \sup_{|u|=1, u \in P} \sup_{x \in V} \int_{-1}^1 G_V(x, y) u^p(y) dy \\ &\leq \sup_{x \in V} \int_{-1}^1 G_V(x, y) dy < \infty. \end{aligned}$$

Hence  $B : P \rightarrow P$  satisfies (7). Recall that the operator  $B$  is compact (see Preliminaries). Since (5) is also satisfied Theorem 1.1 gives that there exists at least two solutions in  $P$  of

$$u = B(u) + u_0.$$

This equation may be rewritten as

$$u = G_V(u^p + h).$$

Lemma 5.3 in [7] implies that the solution of this equation is a weak solution of (8)-(9), which turns out due to the classical bootstrap argument that it is a classical one if we assume the function  $h$  to be Hölder regular of order  $\gamma > 2 - \alpha$ , cf. [25]. So we finally proved that there exists at least two solutions of (8)-(9).  $\square$

The global solvability of some related problem under different conditions guaranteeing the integral operator to be a global diffeomorphism was considered in [9].

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