

# NEUMANN HEAT KERNEL MONOTONICITY

R. BAÑUELOS\*, T. KULCZYCKI\*\* AND B. SIUDEJA\*

ABSTRACT. We prove that the diagonal of the transition probabilities for the  $d$ -dimensional Bessel processes on  $(0, 1]$ , reflected at 1, which we denote by  $p_R^N(t, r, r)$ , is an increasing function of  $r$  for  $d > 2$  and that this is false for  $d = 2$ .

## 1. INTRODUCTION

The following conjecture of Richard Laugesen and Carlo Morpurgo arose, as communicated to us by R. Laugesen, in connection with their work in [12] on conformal extremals of zeta functions of eigenvalues under Neumann boundary conditions. While this may be the first time the conjecture appears in print, the problem seems to be well-known.

**Conjecture 1.1.** *Let  $\mathbb{B}$  be the unit ball in  $\mathbb{R}^d$ ,  $d \geq 2$ , and let  $p_{\mathbb{B}}^N(t, x, y)$  be the heat kernel for the Laplacian in  $\mathbb{B}$  with Neumann boundary conditions. Equivalently,  $p_{\mathbb{B}}^N(t, x, y)$  gives the transition probabilities for the Brownian motion in  $\mathbb{B}$  with normal reflection on the boundary. Fix  $t > 0$ . The (radial) function  $p_{\mathbb{B}}^N(t, x, x)$  increases as  $|x|$  increases to 1. That is, for all  $t > 0$ ,*

$$(1.1) \quad p_{\mathbb{B}}^N(t, x_1, x_1) < p_{\mathbb{B}}^N(t, x_2, x_2),$$

whenever  $0 \leq |x_1| < |x_2| \leq 1$ .

Of course, the same conjecture makes sense for  $d = 1$ . For this, see Remark 5.4 in §5.

We should observe here that for the Dirichlet heat kernel in  $\mathbb{B}$ , the opposite inequality is true. That is, the diagonal of the Dirichlet heat kernel decreases as the point moves toward the boundary (see Proposition 5.2 in §5 below).

The conjecture is closely related to the *hot-spots Conjecture* of Jeff Rauch which asserts that the maxima and minima of any eigenfunction  $\varphi_1$  corresponding to the smallest positive Neumann eigenvalue  $\mu_1$  of a convex planar domain are attained at the boundary, and only at the boundary, of the domain. Indeed, if we denote the volume of the unit ball  $\mathbb{B}$  in  $\mathbb{R}^d$  by  $\omega_d$ , the

---

<sup>0</sup>The first and third authors were supported in part by NSF grant # 0603701-DMS.

The second author was supported in part by KBN Grant 1 P03A 020 28

2000 Mathematics Subject Classification: Primary 31B05, 60J45.

Key words and phrases: Neumann Heat kernels, reflected Brownian motion, random walks, Bessel processes.

eigenfunction expansion of the heat kernels gives that

$$(1.2) \quad p_{\mathbb{B}}^N(t, x, x) \approx \frac{1}{\omega_d} + e^{-\mu_1 t} |\varphi_1(x)|^2,$$

and this is uniform in  $x$  for  $t$  large (see [15]). We refer the reader to [2], [8], [4], and references therein, for more on the *hot-spots* conjecture and for the use of heat kernel expansions and transition probabilities for that problem. Of course, for the unit ball the *hot-spots* conjecture follows easily from the explicit expression of  $\varphi_1$  as a Bessel function. However, a more general Laugesen-Morpurgo Conjecture can be stated where the connection to the *hot-spots* conjecture is more meaningful, see Conjecture 5.1 below. Surprisingly the *hot-spots* conjecture is open even for an arbitrary triangle in the plane. But perhaps even more surprising is the fact that the Laugesen-Morpurgo conjecture is open even for the unit disk in the plane.

The Neumann heat kernel  $p_{\mathbb{B}}^N(t, x, y)$  gives the transition probabilities for the Brownian motion reflected on the boundary of the ball and hence the use of probability for this problem (just as in the case of the *hot-spots* conjecture) is very natural. The Brownian motion in the ball has a skew symmetric decomposition in terms of a Bessel processes (the radial part) and spherical Brownian motion running with a clock that depends on the Bessel processes (see for example [6]). That is, let  $W_t^{\mathbb{B}}$  be reflected  $d$ -dimensional Brownian motion (RBM) in the ball  $\mathbb{B}$  and let  $R_t$  be the  $d$ -dimensional Bessel process in the interval  $I = (0, 1]$  reflected at 1. Then  $R_t$  is the radial part of  $W_t^{\mathbb{B}}$ . That is,  $R_t = |W_t^{\mathbb{B}}|$ . Let  $p_I^R(t, r, \rho)$  be the transition probabilities for  $R_t$  in the interval  $I = (0, 1]$ . We will often refer to this as the heat kernel for  $|W_t^{\mathbb{B}}|$ . The main result of this paper is the following

**Theorem 1.2.** *Suppose  $d > 2$ . Fix  $t > 0$ . The function  $p_I^R(t, r, r)$  is increasing in  $r$ . That is,*

$$(1.3) \quad p_I^R(t, r_1, r_1) < p_I^R(t, r_2, r_2)$$

for  $0 < r_1 < r_2 \leq 1$ . This monotonicity property fails if  $d = 2$ .

It is known (see [14], page 415) that the transition probabilities (heat kernel)  $q(t, r, \rho)$  for the free 2-dimensional Bessel process is given by

$$(1.4) \quad q(t, r, \rho) = t^{-1} \rho e^{-\frac{r^2 + \rho^2}{2t}} I_0\left(\frac{r\rho}{t}\right),$$

where  $I_0$  is the modified Bessel function of order 0. The function  $q(t, r, r)$  is not increasing. In fact, as  $t \rightarrow 0$  this function has a ‘‘tall bump’’ moving toward 0. Since the reflected process is equal to the free process before the first reflection, it seems reasonable to expect that  $p_I^R(t, r, r)$  is not increasing for small values of  $t$  when  $d = 2$ . A rigorous proof of this fact will be given below. On the other hand, the function  $q(t, r, r)$  is non-decreasing for  $d$ -dimensional Bessel processes when  $d \geq 3$  and therefore one may expect the monotonicity property to hold for  $p_I^R(t, r, r)$  for such  $d$  and Theorem 1.2 shows that this is indeed the case.

Our strategy in this paper is to replace the reflected Bessel process by a random walk and obtain the result for this walk. We then show that under the appropriate scale the random walk converges to the reflected Bessel process.

The paper is organized as follows. In §2 we introduce the random walks and prove the analogue of Theorem 1.2 for these. §3 contains the proof of the convergence of these random walks to their continuous counterparts and §4 gives the proof of theorem 1.2. Finally, the last section contains some conjectures related to our result and illuminates the connection between the Rauch *hot-spots* Conjecture and the Laugesen–Morpurgo Conjecture further.

## 2. THE RANDOM WALK

In this section we introduce the random walk which we will use later to approximate the reflected Bessel process. We will use the following notation. For any  $x \in \mathbb{R}^d$ ,  $|x|$  denote the length of the vector  $x$ . For  $d \geq 3$  and  $x \neq 0$ , set

$$U(x) = \frac{|x| + 1}{|x|}x$$

and

$$D(x) = \frac{|x| - 1}{|x|}x.$$

Consider the two sets

$$C(x) = \left\{ y \in \mathbb{R}^d : |y| = |x| + 1 \text{ and } y - D(x) \perp x \right\},$$

and

$$S = \left\{ y \in \mathbb{R}^d : |y| \in \mathbb{N} \text{ and } |y| \geq \frac{d}{2} - 1 \right\}.$$

Note that  $C(x)$  is a  $(d - 2)$ -dimensional sphere with center at  $D(x)$  and orthogonal to  $x$ . We now define two random walks as follows.

**Definition 2.1.** *Let  $X_n$  be a random walk on  $S$  with the following transition probabilities*

- (1)  $p(1, x, U(x)) = \frac{1}{2}$ ,
- (2)  $p(1, x, D(x)) = \frac{1}{2} - \frac{d-1}{4|x|}$ , for  $|x| \geq \frac{d-1}{2}$ ,
- (3)  $p(1, x, A) = \frac{d-1}{4|x|}\mu_x(A)$ , where  $A \subset C(x)$  and  $\mu_x$  a uniform probability measure on  $C(x)$ , for  $|x| \geq \frac{d-1}{2}$ ,
- (4)  $p(1, x, A) = \frac{1}{2}\mu_x(A)$ , where  $A \subset C(x)$  and  $\mu_x$  a uniform probability measure on  $C(x)$ , for even  $d$  and  $|x| = \frac{d-2}{2}$ .

Observe that if  $d = 2k + 1$ , then  $|x| \geq k$  and  $p(1, x, D(x)) = 0$  if  $|x| = k$ . If  $d = 2k$ , then  $|x| \geq k - 1$  and  $p(1, x, D(x)) = \frac{1}{4k}$  if  $|x| = k$ . Hence, we need the additional points  $|x| = k - 1$  as in the last case in the definition.

We will also be concerned with the radial part of the above random walk.

**Definition 2.2.** *Let  $Y_n = |X_n|$ . Then  $Y_n$  has the following transition probabilities*

- (1)  $p(1, m, m + 1) = \frac{1}{2} + \frac{d-1}{4m}$ , for  $m \geq \frac{d-1}{2}$ ,
- (2)  $p(1, m, m + 1) = 1$ , for even  $d$  and  $m = \frac{d}{2} - 1$ ,
- (3)  $p(1, m, m - 1) = \frac{1}{2} - \frac{d-1}{4m}$ , for  $m \geq \frac{d-1}{2}$ .

The above are the “free” random walks. For our purpose, we define the reflected versions of these walks.

**Definition 2.3.** *Let  $X_n^N$  be the random walk on  $S \cap \{|x| \leq N\}$  with transition probabilities  $p_N(1, x, y) = p(1, x, y)$  for  $|x| < N$  and*

- (1)  $p_N(1, x, x) = \frac{1}{2} + \frac{d-1}{4|x|}$  for  $|x| = N$ ,
- (2)  $p_N(1, x, D(x)) = \frac{1}{2} - \frac{d-1}{4|x|}$  for  $|x| = N$

and denote by  $Y_n^N = |X_n^N|$  its radial part.

We will use  $X_n^N$  and  $Y_n^N$  to approximate  $W_t^{\mathbb{B}}$  and  $R_t = |W_t^{\mathbb{B}}|$ , respectively. We will now show that  $p_N(n, m, m)$ , the transition probabilities for the random walk  $Y_n^N$ , are increasing with  $m$  for any fixed  $n$ . The following property will be useful for this purpose.

**Definition 2.4.** *We say, that a random walk has the nondecreasing loop property if for any  $m$  its transition probabilities have the property that*

$$p(1, m, m + 1)p(1, m + 1, m)$$

is nondecreasing with  $m$ . In case of a reflected walk we also require that  $p_N(1, N, N)^2$  (the loop at the reflection point) be larger than or equal to

$$p(1, m, m + 1)p(1, m + 1, m),$$

for every  $m < N$ .

**Lemma 2.5.** *The random walks  $Y_n$  and  $Y_n^N$  have the nondecreasing loop property.*

*Proof.* In the case of  $Y_n$  we have for  $m \geq (d-1)/2$

$$\begin{aligned}
 p(1, m, m+1)p(1, m+1, m) &= \left(\frac{1}{2} + \frac{d-1}{4m}\right) \left(\frac{1}{2} - \frac{d-1}{4(m+1)}\right) \\
 &= \frac{(2m+d-1)(2m+2-d+1)}{16m(m+1)} \\
 (2.1) \qquad \qquad \qquad &= \frac{4(m^2+m) - (d-1)(d-3)}{16(m^2+m)} \\
 &= \frac{1}{4} - \frac{(d-1)(d-3)}{16(m^2+m)}.
 \end{aligned}$$

Thus the left hand side is nondecreasing for  $m \geq \frac{d-1}{2}$ . When  $d=3$  this quantity is constant.

Next we take  $m = \frac{d}{2} - 1$  for  $d$  even. In this case

$$(2.2) \qquad p\left(1, \frac{d}{2} - 1, \frac{d}{2}\right) p\left(1, \frac{d}{2}, \frac{d}{2} - 1\right) = \frac{1}{2d}.$$

From the general case

$$\begin{aligned}
 p\left(1, \frac{d}{2}, \frac{d}{2} + 1\right) p\left(1, \frac{d}{2} + 1, \frac{d}{2}\right) &= \frac{1}{4} - \frac{(d-1)(d-3)}{4d^2 + 8d} \\
 &= \frac{6d-3}{4d^2 + 8d} \\
 &> \frac{2d+4}{4d^2 + 8d} = \frac{1}{2d},
 \end{aligned}$$

since  $d \geq 4$ . This completes the proof that  $Y_n$  has the non-decreasing loop property.

In the case of  $Y_n^N$  we are left with reflection point loop, i.e.  $p(1, N, N)^2$ . But this is larger than  $1/4$ , hence we have a nondecreasing loop property for  $Y_n^N$ .  $\square$

**Proposition 2.6.** *Fix  $n$ . Then  $p_N(n, m, m)$  is increasing in  $m$ .*

*Proof.* To prove this we fix  $n$  and consider each possible path from  $m$  to  $m$  in  $n$  steps. The proof will be completed if for each of them we can find a unique path from  $m+1$  to  $m+1$  in  $n$  steps that has a larger probability. Towards this end, let  $P_m = \{m = l_1, l_2, \dots, l_{n-1}, l_n = m\}$  be a path for  $Y^N$ . Let  $k_1 = \inf\{k : l_k = N\}$  and  $k_2 = \sup\{k : l_k = N\}$ . If this path never touches  $N$  then we can take the path  $\mathbf{P}_{m+1} = \{m+1 = l_1+1, l_2+1, \dots, l_n+1 = m+1\}$ . Since both paths start and end at the same point, they are (possibly after rearranging) a sequences of loops. By the nondecreasing loop property for  $Y^N$  proved in Lemma 2.5, the probability for the path  $P_{m+1}$  is larger.

Up to now we have only used those paths starting at  $m+1$  that do not remain at  $N$ . That is, those paths which move to  $N-1$  immediately after hitting  $N$ . This is true, since all the paths  $P_m$  we considered above never

touched  $N$ , their maximum can be  $N - 1$  and the walk has to move to  $N - 2$  from there.

Suppose that  $P_m$  hits  $N$  at time  $k_1 < N$ . Then  $k_2 < N$ . Let  $P_{m+1} = \{m+1 = l_1+1, l_2+1, \dots, l_{k_1-1}+1, l_{k_1}, \dots, l_{k_2}, l_{k_2+1}+1, \dots, l_n+1 = m+1\}$ . The idea is to shift the parts of the path before and after hitting  $N$  for the first and last time. We have to show that such correspondence of the paths is one-to-one.

We have to look at the parts of  $P_m$  before, after and in between the hitting times, separately. First notice that for  $P_{m+1}$  the step from  $k_1 - 1$  to  $k_1$  is the first time the path remains at  $N$ . Similarly  $k_2$  to  $k_2 + 1$  is the last time the path remains in  $N$ . Hence if two different paths  $P_m$  have different times  $k_1$  or  $k_2$ , then the shifted paths  $P_{m+1}$  will also be different. Hence the only possible paths with the same corresponding shifted paths must have the same hitting times  $k_1$  and  $k_2$ .

Note that it is important in this proof that the walk cannot stay at any point other than the reflection point  $N$ . If we allow  $p(1, N - 1, N - 1)$  to be non-zero, then the path  $P_{m+1}$  obtained from the path  $P_m$  that never touches  $N$  may remain at the point  $N$  even before the time  $k_1$ . This would invalidate the above reasoning.

If two paths  $P_m$  are different at any shifted point (before  $k_1$  or after  $k_2$ , then the same is true for the corresponding paths  $P_{m+1}$ . Finally, if both  $k_1$  and  $k_2$  are the same for two different paths  $P_m$  and they are the same before  $k_1$  and after  $k_2$ , then there must be a difference between  $k_1$  and  $k_2$ . But this part of those paths is not changed in  $P_{m+1}$ . Therefore the correspondence between  $P_m$  and  $P_{m+1}$  is one-to-one.

The last thing to check is that the probability of the corresponding  $P_{m+1}$  path is larger. Since the part between  $k_1$  and  $k_2$  is exactly the same, we can disregard it. What is left is just a sequence of loops (after rearrangement). Hence by the nondecreasing loop property (Lemma 2.5), this completes the proof.  $\square$

As a corollary to the above argument we get

**Corollary 2.7.** *Fix  $n$ . For any  $m' < m$ ,  $p > 0$  and  $m + p \leq N$  we have  $p_N(n, m, m') \leq p_N(n, m + p, m' + p)$ .*

The proof is almost identical. The only difference is that each path can be decomposed into loops plus additional transitions from  $m$  to  $m'$  (all of them toward 0). But since  $p_N(1, n, n - 1) = \frac{1}{2} - \frac{d-1}{4n}$  is increasing with  $n$ , the path shifted by  $p$  will have larger probability.  $\square$

### 3. CONVERGENCE

**Proposition 3.1.** *The sequence  $\{\frac{1}{N} X_{[N^2 t]}^N\}$  converges weakly to the reflected Brownian motion  $W_t^{\mathbb{B}}$  as  $N \rightarrow \infty$ .*

The proof of this fact is essentially the same as the convergence proof in [10]. First we need to establish the existence of a weak limit of the

process  $Z_t^N$  that interpolates  $\frac{1}{N}X_{[N^2t]}^N$  linearly. That is, of the continuous process that equals  $\frac{1}{N}X_{[N^2t]}^N$  at the times of the jumps of the process and is linear in between. The process  $Z_t^N$  converges weakly by Hölder continuity and Prohorov's theorem (see [10].) Our main goal here is to identify this weak limit as RBM on  $\mathbb{B}$ . To accomplish this we use the submartingale characterization of the reflected Brownian motion (see introduction in [16]). More precisely, the RBM in  $\mathbb{B}$  is the only stochastic process starting from  $x \in \mathbb{B}$  such that for any  $f \in C_b^2(\mathbb{B})$  with positive normal derivative at each point of the boundary of  $\mathbb{B}$ , the process

$$(3.1) \quad f(W_t^{\mathbb{B}}) - \int_0^t \Delta f(W_s^{\mathbb{B}}) ds$$

is a submartingale. Hence, to prove that  $Z_t^N \rightarrow W_t^{\mathbb{B}}$  weakly, it is enough to show that

$$(3.2) \quad \liminf_{N \in \mathbb{N}} E \left( f(Z_t^N) - f(Z_s^N) - \frac{1}{2} \int_s^t \Delta f(Z_u^N) du \right) \geq 0,$$

for all such functions  $f$ . Here and in the sequel,  $\Delta$  denotes the Laplacian in  $\mathbb{R}^d$ .

First we will calculate an expectation of the single jump of  $\frac{1}{N}X_{[N^2t]}^N$ . Note that by the definition,  $Z_t^N = \frac{1}{N}X_{[N^2t]}^N$  at the jump times.

**Lemma 3.2.** *Let  $u_n = n/N^2$  be the points where the process  $\frac{1}{N}X_{[N^2t]}^N$  makes its jumps. Then*

$$(3.3) \quad \begin{aligned} \mathbf{E}^x \left( f(Z_{u_{n+1}}^N) - f(Z_{u_n}^N) \right) &= \mathbf{E}^x \left( \frac{1}{2N^2} \Delta f(Z_{u_n}^N) + o(N^{-2}) \right. \\ &\quad \left. + O(N^{-2}) 1_{\{|Z_{u_n}^N| = \frac{d-2}{2N}\}} \right. \\ &\quad \left. + (-c_N \partial_1 f(Z_{u_n}^N) + O(N^{-2})) 1_{\{|Z_{u_n}^N| = 1\}} \right), \end{aligned}$$

where  $\partial_1 f$  denotes the outer normal derivative of  $f$  on  $\{y : |y| = |x|\}$ .

*Proof.* Let  $A(x) = \mathbf{E}^{Nx} (f(\frac{1}{N}X_1^N) - f(x))$ . Note that if the starting point for the process  $Z_{u_n}^N$  is  $x$ , then the corresponding starting point of  $X_n^N$  is  $Nx$ . By the strong Markov property for  $X^N$  and the definition of  $Z_t^N$ ,

$$(3.4) \quad \begin{aligned} \mathbf{E}^x \left( f(Z_{u_{n+1}}^N) - f(Z_{u_n}^N) \right) &= \mathbf{E}^{Nx} \left( f \left( \frac{1}{N}X_{n+1}^N \right) - f \left( \frac{1}{N}X_n^N \right) \right) \\ &= \mathbf{E}^{Nx} \mathbf{E}^{X_n^N} \left( f \left( \frac{1}{N}X_1^N \right) - f \left( \frac{1}{N}X_0^N \right) \right) \\ &= \mathbf{E}^{Nx} A \left( \frac{1}{N}X_n^N \right) = \mathbf{E}^x A(Z_{u_n}^N). \end{aligned}$$

Let  $\mu_x$  be the uniform probability measure on  $C(Nx)/N$ . For  $1 > |x| \geq \frac{d-1}{2N}$ ,  $|x| = k/N$  (the states of the rescaled process) and for any function

$f \in C_b^2$ ,

$$(3.5) \quad A(x) = \frac{1}{2}f\left(\frac{U(Nx)}{N}\right) + \left(\frac{1}{2} - \frac{d-1}{4N|x|}\right)f\left(\frac{D(Nx)}{N}\right) + \frac{d-1}{4N|x|} \int_{\frac{C(Nx)}{N}} f(y) d\mu_x(y) - f(x),$$

Let  $\partial_1$  denotes the outer normal derivative to the sphere  $\{y : |y| = |x|\}$  at the point  $x$ . Let also  $\partial_{11}^2 = \partial_1 \partial_1$ . We have

$$(3.6) \quad \begin{aligned} A(x) &= \frac{1}{2} \left( \frac{1}{N} \partial_1 f(x) + \frac{1}{2N^2} \partial_{11}^2 f(x') \right) \\ &+ \left( \frac{1}{2} - \frac{d-1}{4N|x|} \right) \left( -\frac{1}{N} \partial_1 f(x) + \frac{1}{2N^2} \partial_{11}^2 f(x'') \right) \\ &+ \frac{d-1}{4N|x|} \int_{C(Nx)/N} [(y-x) \cdot \nabla] f(x) + \frac{1}{2} [(y-x) \cdot \nabla]^2 f(z(y)) d\mu_x(y) \\ &= \frac{1}{2N^2} \partial_{11}^2 f(x') - \frac{d-1}{4N|x|} \left( -\frac{1}{N} \partial_1 f(x) + \frac{1}{2N^2} \partial_{11}^2 f(x'') \right) \\ &+ \frac{d-1}{4N|x|} \int_{C(Nx)/N} [(y_1 - x_1) \partial_1] f(x) + \frac{1}{2} [(y-x) \cdot \nabla]^2 f(z(y)) d\mu_x(y), \end{aligned}$$

since  $C(x)$  is a sphere on  $d-1$  dimensional hyperplane orthogonal to  $x$  and centered in  $D(x)$ . We also have  $y_1 - x_1 = -1/N$  hence the first order terms cancel and we get

$$(3.7) \quad \begin{aligned} A(x) &= \frac{1}{2N^2} \partial_{11}^2 f(x') - \frac{d-1}{8N^3|x|} \partial_{11}^2 f(x'') \\ &+ \frac{d-1}{8N|x|} \int_{C(Nx)/N} [(y-x) \cdot \nabla]^2 f(z(y)) d\mu_x(y). \end{aligned}$$

For  $x \in B(0,1)$ , the function  $f$  has uniformly continuous second order derivatives. Hence the error in the Taylor expansion is uniformly bounded. If we denote the derivatives in the directions tangent to the sphere  $\{y : |y| = |x|\}$  at  $x$  by  $\partial_i$ ,  $i \geq 2$ , and  $\partial_1$  as before denotes the outer normal derivative to the sphere  $\{y : |y| = |x|\}$ , then

$$(3.8) \quad \begin{aligned} &\int_{C(Nx)/N} [(y-x) \cdot \nabla]^2 f(z(y)) d\mu_x(y) \\ &= \int_{C(Nx)/N} [(y-x) \cdot \nabla]^2 f(x) + o(|y-x|^2) d\mu_x(y) \\ &= \int_{C(Nx)/N} \sum_{i=1}^n (x_i - y_i)^2 \partial_{ii}^2 f(x) d\mu_x(y) + o\left(\frac{|x|}{N}\right), \end{aligned}$$

since  $|y-x|^2 = 4(|x| + 1/N)/N$  and mixed derivatives disappear due to the symmetry of  $C(Nx)/N$ .



If  $i = 1$  in the above sum, then  $(x_1 - y_1)^2 = 1/N^2$ . For each  $i \geq 2$  the integral has the same value due to the symmetry of  $C(Nx)/N$ . Also

$$(3.9) \quad \sum_{i=2}^n (x_i - y_i)^2 = \frac{4|x|}{N}.$$

Hence

$$(3.10) \quad \begin{aligned} A(x) &= \frac{1}{2N^2} \partial_{11}^2 f(x) + o(N^{-2}) - \frac{d-1}{8N^3|x|} \partial_{11}^2 f(x) + o(N^{-3}|x|^{-1}) \\ &+ \frac{d-1}{8N|x|} \left[ \frac{1}{N^2} \partial_{11}^2 f(x) + \sum_{i=2}^d \frac{4|x|}{N(d-1)} \partial_{ii}^2 f(x) + o\left(\frac{|x|}{N}\right) \right] \\ &= \frac{1}{2N^2} \Delta f(x) + o(N^{-2}), \end{aligned}$$

since  $2N|x| \geq d-1$ .

Now we have to consider two special cases. First suppose  $d$  is even and  $|x| = \frac{d-2}{2N}$ . Then

$$(3.11) \quad A(x) = O(N^{-2}) = \frac{1}{2N^2} \Delta f(x) + O(N^{-2}).$$

Note that the error is uniform, just like in the first case.

The remaining case is the reflection circle. That is, the points  $|x| = 1$ . These points correspond to the times when  $|X_{[N^2t]}^N| = N$ . That is, when the walk  $X^N$  is on the boundary. Hence we have to use the transition steps from Definition 2.3. Now,

$$(3.12) \quad \begin{aligned} A(x) &= \left(\frac{1}{2} + \frac{d-1}{4N|x|}\right) f(x) + \left(\frac{1}{2} - \frac{d-1}{4N|x|}\right) f(D(Nx)/N) - f(x) \\ &= \left(-\frac{1}{2} + \frac{d-1}{4N}\right) f(x) + \left(\frac{1}{2} - \frac{d-1}{4N}\right) \left(f(x) - \frac{1}{N} \partial_1 f(x) + O(N^{-2})\right) \\ &= -\frac{1}{N} \left(\frac{1}{2} - \frac{d-1}{4N|x|}\right) \partial_1 f(x) + O(N^{-2}) \\ &= -c_N \partial_1 f(x) + \frac{1}{2N^2} \Delta f(x) + O(N^{-2}). \end{aligned}$$

As before,  $\partial_1$  is the outer normal derivative to the sphere  $\{y : |x| = |y|\}$ . Hence  $-\partial_1$  becomes a normal derivative along the reflection direction if  $x$  is on the boundary ( $|x| = 1$ ). By the definition of  $f$  the first order term above is positive.

Combining all the cases we obtain

$$(3.13) \quad \begin{aligned} A(x) &= \frac{1}{2N^2} \Delta f(x) + o(N^{-2}) + O(N^{-2}) 1_{\{|x| = \frac{d-2}{2N}\}} \\ &+ (-c_N \partial_1 f(x) + O(N^{-2})) 1_{\{|x|=1\}}. \end{aligned}$$

This and (3.4) give the assertion of the lemma.

□

We are now ready to prove (3.2). Summing the expressions from the above lemma over  $s \leq u_n \leq t$  yields

$$\begin{aligned}
\mathbf{E}^x (f(Z_t^N) - f(Z_s^N)) &= \mathbf{E}^x \left( \int_s^t \frac{1}{2} \Delta f(Z_u^N) du \right) \\
&+ \mathbf{E}^x \left( \mathcal{L}_{\frac{d-2}{2N}}^N(s, t) \right) O(N^{-2}) \\
(3.14) \quad &+ \sum_{s \leq u_n \leq t} E^x \left( -c_N \partial_1 f(Z_{u_n}^N) 1_{\{|Z_{u_n}^N|=1\}} \right) \\
&+ \mathbf{E}^x \left( \mathcal{L}_1^N(s, t) \right) O(N^{-2}) + o(1),
\end{aligned}$$

where  $\mathcal{L}_\alpha^N(s, t)$  denotes the local time (number of visits) of  $Z^N$  on the set  $|x| = \alpha$  between times  $s$  and  $t$ . Note that the term involving  $-\partial_1 f$  is always positive, since by the definition  $f$  has a positive derivative in the reflection direction. In order to finish the proof of (3.2), we need the following lemma

**Lemma 3.3.** *Let  $L_y^{N^2} = \#\{n : Y_n^N = y, n \leq N^2\}$  be the local time at  $y$ . Then*

$$(3.15) \quad \mathbf{E}^y(L_N^{N^2}) = o(N^2).$$

Moreover,  $\mathbf{E}^y(L_y^{N^2})$  is increasing with  $y$ .

We need to reduce the local time of the process  $Z_t^N$  to the local time of the process  $Y_n^N$ . Since  $t$  is fixed, the expectation of the local time  $\mathcal{L}_\alpha^N(s, t)$  is comparable (with constant depending on  $t$  but not on  $N$ ) to the expectation of same on the interval 0 to 1. More precisely, by the strong Markov property for any  $\alpha \leq 1$  we have

$$\begin{aligned}
\mathbf{E}^x(\mathcal{L}_\alpha^N(s, t)) &\leq \mathbf{E}^\alpha(\mathcal{L}_\alpha^N(0, t)) = \mathbf{E}^\alpha \left( \mathcal{L}_\alpha^N(0, 1) + \mathbf{E}^{Z_1^N}(\mathcal{L}_\alpha^N(0, t-1)) \right) \\
(3.16) \quad &\leq \mathbf{E}^\alpha \left( \mathcal{L}_\alpha^N(0, 1) + \mathbf{E}^\alpha(\mathcal{L}_\alpha^N(0, t-1)) \right) \\
&= \mathbf{E}^\alpha(\mathcal{L}_\alpha^N(0, 1)) + \mathbf{E}^\alpha(\mathcal{L}_\alpha^N(0, t-1)) \\
&\leq \dots \leq \lceil t \rceil \mathbf{E}^\alpha(\mathcal{L}_\alpha^N(0, 1)),
\end{aligned}$$

where  $\lceil t \rceil$  is the smallest integer bigger or equal to  $t$ .

But, the local time  $\mathcal{L}_\alpha^N(0, 1)$  of  $Z^N$  is the same as the local time  $L_{N\alpha}^{N^2}$  of  $Y^N$ . Hence, by Lemma 3.3 both error terms involving the local time in (3.14) are negligible. This ends the proof of (3.2). Hence the process  $\frac{1}{N} X_{\lfloor N^2 t \rfloor}^N$  converges weakly to  $W_t^B$ . It follows, that  $\frac{1}{N} Y_{\lfloor N^2 t \rfloor}^N$  converges weakly to  $R_t$ . The monotonicity of  $p_R^N(t, x, x)$  will follow from the monotonicity of the approximating random walk, as we shall show in §4 below.

*Proof the Lemma 3.3.* The idea of the proof is based on the following well known facts for the random walk  $Z_n$  on  $\mathbb{Z}$ ,  $Z_0 = 0$ ,  $P(Z_{n+1} = Z_n + 1) =$

$P(Z_{n+1} = Z_n - 1) = 1/2$ . It is a standard fact (which follows from the reflection principle) that the number of paths of length  $2n$  satisfying

$$\{Z_1 > 0, Z_2 > 0, \dots, Z_{2n-1} > 0, Z_{2n} = 0\}$$

is

$$\frac{1}{n} \binom{2n-2}{n-1}.$$

Hence,

$$P\{Z_1 > 0, Z_2 > 0, \dots, Z_{2n-1} > 0, Z_{2n} = 0\} = \frac{1}{n} \binom{2n-2}{n-1} 2^{-2n}.$$

Next we show that  $E^x(L_N^{N^2}) = o(N^2)$ . Of course,  $E^N(L_N^{N^2}) \geq E^x(L_N^{N^2})$  so it is sufficient to show that  $E^N(L_N^{N^2}) = o(N^2)$ . Consider the sequence of stopping times  $R_0 = 0, R_{k+1} = \inf\{m > R_k : Y^N(m) = N\}$ . We have

$$E^N(L_N^{N^2}) = E^N(\max\{k \in \mathbb{N} : R_k \leq N^2\}).$$

We have  $R_k = \sum_{j=1}^k (R_j - R_{j-1})$ .  $\{R_j - R_{j-1}\}_{j=1}^\infty$  is a sequence of i.i.d. random variables with  $R_j - R_{j-1} \stackrel{d}{=} R_1$ . Let  $\{S_i\}_{i=1}^\infty$  be a sequence of i.i.d. random variables with  $S_i \stackrel{d}{=} R_1 = \inf\{m > 0 : Y^N(m) = N\}$  and  $T_i = S_i \wedge 2\lfloor N/4 \rfloor$ . We have

$$\begin{aligned} E^N(L_N^{N^2}) &= E^N(\max\{k \in \mathbb{N} : S_1 + \dots + S_k \leq N^2\}) \\ (3.17) \quad &\leq E^N(\max\{k \in \mathbb{N} : T_1 + \dots + T_k \leq N^2\}). \end{aligned}$$

Note also that  $N - 2\lfloor N/4 \rfloor \geq N/2 \geq 2\lfloor N/4 \rfloor$ . We may and do assume that  $N$  is large enough so that  $2\lfloor N/4 \rfloor > (d-1)/2$ .

Our next aim is to estimate  $E^N T_1$ . Let  $2n \leq 2\lfloor N/4 \rfloor$ . The number of paths of length  $2n$  satisfying

$$\{Y_0^N = N, Y_1^N < N, Y_2^N < N, \dots, Y_{2n-1}^N < N, Y_{2n}^N = N\}$$

is

$$\frac{1}{n} \binom{2n-2}{n-1}.$$

Using Definitions 2.2 and 2.3 and the fact that  $m \geq N - 2n \geq N - 2\lfloor N/4 \rfloor \geq 2\lfloor N/4 \rfloor$ , we obtain

$$\begin{aligned} P^N(S_1 = 2n) &= P^N\{Y_0^N = N, Y_1^N < N, \dots, Y_{2n-1}^N < N, Y_{2n}^N = N\} \\ &\geq \frac{1}{n} \binom{2n-2}{n-1} \left(\frac{1}{2} - \frac{d-1}{8\lfloor N/4 \rfloor}\right)^{2n}. \end{aligned}$$

Observe that

$$\left(\frac{1}{2} - \frac{d-1}{8\lfloor N/4 \rfloor}\right)^{2n} \geq \frac{1}{2^{2n}} \left(1 - \frac{d-1}{4\lfloor N/4 \rfloor}\right)^{2\lfloor N/4 \rfloor} \geq \frac{c}{2^{2n}}.$$

We now adopt the convention that  $c = c(d) > 0$  is a positive constant which can change its value from line to line.

By Stirling formula

$$\begin{aligned} \frac{1}{n} \binom{2n-2}{n-1} \frac{1}{2^{2n}} &= \frac{(2n-2)!}{n((n-1)!)^2 2^{2n}} \\ &\geq \frac{(2n-2)^{2n-2} e^{-(2n-2)} \sqrt{2\pi(2n-2)}}{n(n-1)^{2(n-1)} e^{-2(n-1)} 2\pi(n-1) e^{2/(12(n-1))} 2^{2n}} \\ &\geq \frac{c}{n^{3/2}}. \end{aligned}$$

Hence  $P^N(S_1 = 2n) \geq c/n^{3/2}$ . It follows that

$$E^N T_1 \geq \sum_{n=1}^{\lfloor N/4 \rfloor} 2n P^N(S_1 = 2n) \geq cN^{1/2}.$$

Now we will estimate (3.17). Note that  $T_1 \geq 1$  so  $\max\{k \in \mathbb{N} : T_1 + \dots + T_k \leq N^2\} \leq N^2$ .

Let  $M = \lfloor 2N^2/E^N T_1 \rfloor + 1$  so that  $ME^N T_1 - N^2 \geq N^2$ . Note that  $M \leq cN^{3/2}$ . We have

$$\begin{aligned} E^N(L_N^{N^2}) &\leq E^N(\max\{k \in \mathbb{N} : T_1 + \dots + T_k \leq N^2\}) \\ &= E^N(\max\{k \in \mathbb{N} : T_1 + \dots + T_k \leq N^2; T_1 + \dots + T_M > N^2\}) \\ &\quad + E^N(\max\{k \in \mathbb{N} : T_1 + \dots + T_k \leq N^2; T_1 + \dots + T_M \leq N^2\}) \\ (3.18) \quad &\leq M + N^2 P^N(T_1 + \dots + T_M \leq N^2). \end{aligned}$$

We have

$$\begin{aligned} &P^N(T_1 + \dots + T_M \leq N^2) \\ &\leq P^N(|T_1 + \dots + T_M - ME^N T_1| \geq ME^N T_1 - N^2) \\ (3.19) \quad &\leq \frac{E^N(|T_1 + \dots + T_M - ME^N T_1|^2)}{(ME^N T_1 - N^2)^2} \leq \frac{ME^N T_1^2}{N^4}. \end{aligned}$$

Recall that  $M \leq cN^{3/2}$  and  $T_1 = S_1 \wedge 2\lfloor N/4 \rfloor \leq N$ . It follows that  $E^N T_1^2 \leq N^2$  and the last expression in (3.19) is bounded from above by

$$\frac{cN^{3/2}N^2}{N^4} \leq \frac{c}{N^{1/2}}.$$

By (3.18) we obtain  $E^N(L_N^{N^2}) \leq cN^{3/2}$  which gives  $E^N(L_N^{N^2}) = o(N^2)$ .

The monotonicity for the local times follows from

$$\begin{aligned} \mathbf{E}_x(L_x^{N^2}) &= \mathbf{E}_x\left(\sum_{k=0}^{N^2} 1_{\{Y_k^N = x\}}\right) = \sum_{k=0}^{N^2} \mathbf{P}_x(Y_k^N = x) = \sum_{k=0}^{N^2} p_N(k, x, x) \\ &\leq \sum_{k=0}^{N^2} p_N(k, x+1, x+1) = \mathbf{E}_{x+1}(L_{x+1}^{N^2}), \end{aligned}$$

where the inequality follows from the heat kernel monotonicity obtained in Section 2.  $\square$

## 4. PROOF OF THEOREM 1.2

First we give the proof that if  $d = 2$  then  $p_R^N(t, r, r)$  is not increasing for small enough times  $t$ .

Let  $P \subset \mathbb{R}^2$  be a convex polygon and denote its Neumann heat kernel by  $p_P^N(t, x, y)$ . It is proved in [9] that

$$(4.1) \quad \lim_{t \rightarrow 0} \frac{p_P^N(t, x, y)}{p(t, x, y)} = 1$$

uniformly in  $x, y \in P$ , where  $p$  denotes the heat kernel of the free Brownian motion in  $\mathbb{R}^2$ . In addition, [9], also proves that if  $D_1$  is a convex domain whose closure,  $\overline{D_1}$ , is contained in the convex domain  $D_2$ , then there exists a  $t_0$  sufficiently small such that

$$(4.2) \quad p_{D_2}^N(t, x, y) \leq p_{D_1}^N(t, x, y),$$

for all  $x, y \in D_1$  and  $0 < t < t_0$ , where  $t_0$  depends only on the distance between  $\partial D_1$  and  $\partial D_2$ . By taking two polygons  $P_1$  and  $P_2$  such that  $\overline{P_1} \subset \mathbb{I} \subset \overline{P_2}$  and combining (4.1) with (4.2) we see that

$$(4.3) \quad \lim_{t \rightarrow 0} \frac{p_{\mathbb{I}}^N(t, x, y)}{p(t, x, y)} = 1$$

uniformly in  $|x| < \frac{1}{2}$  and  $|y| < \frac{1}{2}$ .

Since for any  $x \in \mathbb{I}$ ,

$$(4.4) \quad P^x\{|W_t^{\mathbb{I}}| \in (a, b)\} = \int_a^b \int_0^{2\pi} \rho p_{\mathbb{I}}^N(t, x, \rho e^{i\theta}) d\theta$$

and the reflected Bessel process is the radial part of the reflected Brownian motion, we have

$$(4.5) \quad p_I^R(t, r, r) = \int_0^{2\pi} r p_{\mathbb{I}}^N(t, r, r e^{i\theta}) d\theta.$$

and similarly,

$$(4.6) \quad q(t, r, r) = \int_0^{2\pi} r p(t, r, r e^{i\theta}) d\theta.$$

Let  $\varepsilon > 0$  and  $r, \rho < 1/2$ . Using (4.3) we can pick  $t(\varepsilon)$  such that for  $t < t(\varepsilon)$

$$(4.7) \quad (1 - \varepsilon)p(t, r, r e^{i\theta}) \leq p_{\mathbb{I}}^N(t, r, r e^{i\theta}) \leq (1 + \varepsilon)p(t, r, r e^{i\theta})$$

$$(4.8) \quad (1 - \varepsilon)p(t, \rho, \rho e^{i\theta}) \leq p_{\mathbb{I}}^N(t, \rho, \rho e^{i\theta}) \leq (1 + \varepsilon)p(t, \rho, \rho e^{i\theta}).$$

From this and the integral formulas for the Bessel heat kernels above

$$(4.9) \quad (1 - \varepsilon)q(t, r, r) \leq p_I^R(t, r, r) \leq (1 + \varepsilon)q(t, r, r)$$

$$(4.10) \quad (1 - \varepsilon)q(t, \rho, \rho) \leq p_I^R(t, \rho, \rho) \leq (1 + \varepsilon)q(t, \rho, \rho).$$

From (1.4) we have

$$(4.11) \quad q(t, r\sqrt{t}, r\sqrt{t}) = t^{-1/2} r e^{-r^2} I_0(r^2)$$

Set  $\Phi_0(r) = r e^{-r^2} I_0(r^2)$ . Using tables of the Bessel functions one can check that  $\Phi_0(1) \approx 0.4657$  and that  $\Phi_0(2) \approx 0.4140$ . Hence  $\Phi_0$  is not nondecreasing. Let  $r < \rho$  be such that  $\Phi_0(r) > \Phi_0(\rho)$ . This is the same as

$$q(t, r\sqrt{t}, r\sqrt{t}) > q(t, \rho\sqrt{t}, \rho\sqrt{t}).$$

Pick  $\varepsilon$  small enough to have

$$(4.12) \quad (1 - \varepsilon)\Phi_0(r) > (1 + \varepsilon)\Phi_0(\rho).$$

Now for any  $t$  we have

$$(4.13) \quad (1 - \varepsilon)q(t, r\sqrt{t}, r\sqrt{t}) > (1 + \varepsilon)q(t, \rho\sqrt{t}, \rho\sqrt{t}).$$

Set  $r_1 = r\sqrt{t}$  and  $r_2 = \rho\sqrt{t}$  so that  $r_1 < r_2$ . If we take  $t$  small enough to have  $t < t(\varepsilon)$  and  $r_2 < \frac{1}{2}$ , it follows from (4.9) and (4.10) that

$$(4.14) \quad p_I^R(t, r_1, r_1) \geq (1 - \varepsilon)q(t, r_1, r_1) > (1 + \varepsilon)q(t, r_2, r_2) \geq p_I^R(t, r_2, r_2).$$

This completes the proof of Theorem 1.2 when  $d = 2$ .

Now we turn to the case  $d \geq 3$ . Fix  $x$  with  $r = |x| < 1$ . Let  $f_\varepsilon(x) = \chi_{[r-\varepsilon, r]}$ ,  $\varepsilon > 0$ . We have

$$(4.15) \quad \mathbf{E}^x \left( f_\varepsilon \left( \frac{1}{N} Y_{[N^2 t]}^N \right) \right) = \mathbf{P}^x \left( \frac{1}{N} Y_{[N^2 t]}^N \in [r - \varepsilon, r] \right).$$

The event above consists of transitions from  $r$  to some points to the left of  $r$ . Hence by Corollary 2.7 this probability is increasing in  $r$ . By the weak convergence of  $\left( \frac{1}{N} Y_{[N^2 t]}^N \right)$  to  $(R_t^N)$ , we have

$$(4.16) \quad \mathbf{E}^x \left( f_\varepsilon \left( \frac{1}{N} Y_{[N^2 t]}^N \right) \right) \rightarrow \mathbf{E}^x (f_\varepsilon(R_t^N)) = \mathbf{P}^x (R_t^N \in [r - \varepsilon, r]).$$

Since the limit of increasing functions is nondecreasing, we have that for arbitrary  $\varepsilon$ ,

$$(4.17) \quad \int_{r_1 - \varepsilon}^{r_1} p_I^R(t, r_1, \rho) d\rho \leq \int_{r_2 - \varepsilon}^{r_2} p_I^R(t, r_2, \rho) d\rho, \text{ if } r_1 < r_2.$$

Suppose that  $p_I^R(t, r_1, r_1) > p_I^R(t, r_2, r_2)$  for some  $r_1 < r_2$ . By the continuity of the heat kernel there exists  $\varepsilon > 0$  such that

$$(4.18) \quad \inf_{\rho \in [r_1 - \varepsilon, r_1]} p_I^R(t, r_1, \rho) > \sup_{\rho \in [r_2 - \varepsilon, r_2]} p_I^R(t, r_2, \rho)$$

Hence,

$$(4.19) \quad \int_{r_1 - \varepsilon}^{r_1} p_I^R(t, r_1, \rho) d\rho > \int_{r_2 - \varepsilon}^{r_2} p_I^R(t, r_2, \rho) d\rho.$$

But this contradicts (4.17). Thus  $p_I^R(t, r, r)$  is nondecreasing in  $r$ .

For any  $t > 0$ , the function  $p_I^R(t, r, r)$  is a real analytic function of  $r \in [\varepsilon, 1]$ , for any  $\varepsilon > 0$ , since it is the diagonal of the heat kernel of an operator with real analytic coefficients. Thus, if it is nondecreasing then it must be strictly increasing. This completes the proof of Theorem 1.2.  $\square$

## 5. FURTHER REMARKS

As mentioned above, the Laugesen–Morpurgo conjecture implies Rauch’s *hot-spots* conjecture for the disk. Of course, as already also mentioned this is a trivial observation since for the disk the Neumann eigenfunctions are all explicitly known (Bessel functions) and the *hot-spots* conjecture is trivial by “inspection”. This observation, however, leads to a more general problem for planar convex domains where the connection to the *hot-spots* conjecture is more meaningful.

**Conjecture 5.1.** *Suppose  $\Omega$  is a bounded convex domain in the plane which is symmetric with respect to the  $x$ -axis. Let  $p_\Omega^N(t, z, w)$  be the Neumann heat kernel for  $\Omega$ . Then  $p_\Omega^N(t, z, z)$  is increasing along hyperbolic radii in  $D$  which intersect the horizontal axis. That is, let  $f : \mathbb{B} \rightarrow \Omega$  be a conformal map of the unit disk  $\mathbb{B} \subset \mathbb{R}^2$  onto the domain  $\Omega$  for which  $f(-1, 1) = \Gamma_f$  is the axis of symmetry of  $\Omega$ . Then for all all  $t > 0$ ,  $p_\Omega^N(t, f(z_1), f(z_1)) < p_\Omega^N(t, f(z_2), f(z_2))$ , where  $z_1 = r_1 e^{i\theta}$ ,  $z_2 = r_2 e^{i\theta}$ ,  $0 < \theta < \pi$  and  $0 < r_1 < r_2 \leq 1$ .*

For a *hot-spots* version, which inspired Conjecture 5.1, we refer the reader to [13] (Theorem 1.1) and [3] (Theorem 3.1).

As mentioned in the introduction, for the Dirichlet heat kernel in  $\mathbb{B}$ , the opposite actually holds. That is, we have the following

**Proposition 5.2.** *Let  $\mathbb{B}$  be the unit ball in  $\mathbb{R}^d$ ,  $d \geq 2$ , and let  $p_{\mathbb{B}}^D(t, x, y)$  be the heat kernel for the Laplacian in  $\mathbb{B}$  with Dirichlet boundary conditions. (Equivalently,  $p_{\mathbb{B}}(t, x, y)$  are the transitions probabilities for the Brownian motion in  $\mathbb{B}$  killed on its boundary of the ball.) Fix  $t > 0$ . The (radial) function  $p_{\mathbb{B}}^D(t, x, x)$  decreasing as  $|x|$  increases to 1. That is, for all  $t > 0$ ,*

$$(5.1) \quad p_{\mathbb{B}}^D(t, x_2, x_2) < p_{\mathbb{B}}^D(t, x_1, x_1),$$

whenever  $0 \leq |x_1| < |x_2| \leq 1$ .

**Remark 5.3.** *People often mention this result and assert that “it is clearly obvious by symmetrization.” However, a proof does not seem to be written down anywhere. We should also mention here that the first attempt for a proof simply based on some type of symmetrization argument, or eigenfunction expansion, seems to rapidly fail. For completeness, we give a short proof here based on the celebrated “log-concavity” results of H. Brascamp and E. Lieb [7].*

*Proof.* Setting  $\Psi(r) = p_{\mathbb{B}}^D(t, x, x)$  for  $|x| = r$  we see that by [7], for any  $0 \leq \lambda \leq 1$ ,

$$(5.2) \quad \Psi(\lambda r_0 + (1 - \lambda)r_1) \geq \Psi(r_0)^\lambda \Psi(r_1)^{(1-\lambda)},$$

for any  $r_0, r_1 \in [0, 1]$ . If  $0 \leq r_1 < r_2 \leq 1$ , we take  $r_0 = 0$  and pick  $\lambda \in (0, 1)$  such that  $(1 - \lambda)r_2 = r_1$ . It follows from (5.2) that

$$(5.3) \quad \Psi(r_1) \geq \Psi(0)^\lambda \Psi(r_2)^{(1-\lambda)}.$$

However, it also follows from the multiple integral re-arrangement inequalities of [7] that

$$\Psi(r) = p_{\mathbb{B}}^D(t, x, x) < p_{\mathbb{B}}^D(t, 0, 0) = \Psi(0),$$

for all  $0 < |x| \leq 1$ . Substituting this into (5.3) proves (5.1) and completes the proof of the proposition.  $\square$

**Remark 5.4.** For the unit interval  $I = (0, 1)$ , the Neumann and Dirichlet heat kernels are given by

$$(5.4) \quad p_I^N(t, x, x) = 1 + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \cos^2(n\pi x)$$

and

$$(5.5) \quad p_I^D(t, x, x) = \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \sin^2(n\pi x),$$

respectively; see [1]. (The “2” is there to normalize the eigenfunctions in  $L^2$ .) Differentiating (5.4) with respect to  $x$  we see that

$$(5.6) \quad \frac{\partial}{\partial x} p_I^N(t, x, x) = -\frac{\partial}{\partial x} p_I^D(t, x, x).$$

However, the same argument used in the proof of Proposition 5.2 above shows that

$$(5.7) \quad p_I^D(t, x_2, x_2) < p_I^D(t, x_1, x_1),$$

whenever  $1/2 \leq x_1 < x_2 \leq 1$ . This together with (5.6) shows that  $p_I^N(t, x, x)$  is increasing on  $(1/2, 1)$  and decreasing on  $(0, 1/2)$  with minimum at  $1/2$ .

**Remark 5.5.** It is interesting to note here that for the interval  $I$ ,

$$(5.8) \quad p_I^N(t, x, x) + p_I^D(t, x, x) = C_t,$$

where  $C_t = 1 + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}$  does not depend on  $x$ .

Since the “log-concavity” result of Brascamp–Lieb holds for all convex domains, the above argument gives the following result for more general convex domains.

**Proposition 5.6.** Let  $\Omega$  be a bounded convex domain in  $\mathbb{R}^2$  which is symmetric relative to the  $x$ -axis. For any  $(x, y) \in \Omega$  we write  $p_{\Omega}^D(t, (x, y), (x, y))$  for the diagonal of the Dirichlet heat kernel in  $\Omega$ . Fix  $x$  and let  $a(x) = \sup\{y > 0 : (x, y) \in \Omega\}$ . The function  $p_{\Omega}^D(t, (x, y), (x, y))$  is decreasing in  $y$  for  $y \in [0, a(x)]$ . By symmetry,  $p_{\Omega}^D(t, (x, y), (x, y))$  is increasing in  $y$  for  $y \in [-a(x), 0]$ .



Motivated by the fact that the heat kernel for Brownian motion conditioned to remain forever in  $\Omega$  satisfies a Neumann-type boundary condition, Bañuelos and Méndez–Hernández proved in [5] an analogue for conditioned Brownian motion of the *hot-spots* result of Jerison and Nadirashvili [11]. Those results and the Laugesen–Morpurgo conjecture motivate the following

**Conjecture 5.7.** *Let  $\varphi_1(x)$  be the ground state eigenfunction for the Laplacian in the unit ball  $\mathbb{B} \subset \mathbb{R}^d$ ,  $d \geq 1$ , with Dirichlet boundary conditions. The radial function*

$$\frac{p_{\mathbb{B}}^D(t, x, x)}{\varphi_1^2(x)}$$

*is increasing as  $|x|$  increases to 1. That is, for all  $t > 0$ ,*

$$(5.9) \quad \frac{p_{\mathbb{B}}^D(t, x_1, x_1)}{\varphi_1^2(x_1)} < \frac{p_{\mathbb{B}}^D(t, x_2, x_2)}{\varphi_1^2(x_2)},$$

*whenever  $0 \leq |x_1| < |x_2| \leq 1$ .*

An appropriate version of this conjecture (similar to Proposition 5.6) motivated by the results in [5] can be formulated for more general convex domains.

#### REFERENCES

- [1] C. Bandle, *Isoperimetric Inequalities and Applications*, Monographs and Studies in Mathematics, Pitman 1980.
- [2] R. Bañuelos and K. Burdzy, *On the “hot spots” conjecture of J. Rauch*. *J. Funct. Anal.* 164 (1999), 1–33.
- [3] R. Bañuelos and M. Pang, *An inequality for potentials and the “hot-spots” conjecture*. *Indiana Univ. Math. J.* 53 (2004), no. 1, 35–47.
- [4] R. Bañuelos, M. Pang and M. Pascu, *Brownian motion with killing and reflection and the “hot spots” problem*. *Prob Theory and Related Fields*, 130 (2004), 56–68.
- [5] R. Bañuelos and P. Méndez-Hernández, *Hot-spots for conditioned Brownian motion*. *Illinois J. Math.* 50 (2006), no. 1-4, 1–32 (electronic).
- [6] R. Bañuelos and R. Smits, *Brownian motion in cones*. *Prob. Th. Rel. Fields*, 108 (1997), 299–319.
- [7] H.L. Brascamp and E.H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*. *J. Funct. Anal.* 22 (1976), 366–389
- [8] K. Burdzy, *The hot spots problem in planar domains with one hole*. *Duke Math. J.* 129 (2005), no. 3, 481–502.
- [9] R. Carmona and W.A. Zheng, *Reflecting Brownian motions and comparison theorems for Neumann heat kernels*. *J. Funct. Anal.* 123 (1994), no. 1, 109–128.
- [10] J. Dubédat, *Reflected planar Brownian motions, intertwining relations and crossing probabilities*. *Ann. Inst. H. Poincaré Probab. Statist.*, 40 (2004), no. 5, 539–552.
- [11] D. Jerison and N. Nadirashvili, *The “hot spots” conjecture for domains with two axes of symmetry*. *J. Amer. Math. Soc.* 13 (2000), no. 4, 741–772 (electronic).
- [12] R. Laugesen and C. Morpurgo, *Extremals for eigenvalues of Laplacians under conformal mapping*. *J. Funct. Anal.* 155 (1998), 64–108.
- [13] M. Pascu, *Scaling coupling of reflecting Brownian motions and the hot spots problem*. *Trans. Amer. Math. Soc.* 354 (2002), no. 11, 4681–4702

- [14] D. Revuz, M. Yor, *Continuous martingales and Brownian motion*. Springer Verlag, 1991.
- [15] R. Smits, Spectral gaps and rates to equilibrium for diffusions in convex domains, *Michigan Math. J.*, **43** (1996), 141–157.
- [16] D. W. Stroock, S.R.S. Varadhan, *Diffusion processes with boundary conditions*, *Comm. Pure Appl. Math.*, **24** (1971), 147–225.

\*DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907-1395

\*\*INSTITUTE OF MATHEMATICS, WROCLAW UNIVERSITY OF TECHNOLOGY, 50-370 WROCLAW, POLAND

*E-mail address:* Bañuelos: `banuelos@math.purdue.edu`

*E-mail address:* Kulczycki: `Tadeusz.Kulczycki@pwr.wroc.pl`

*E-mail address:* Siudeja: `siudeja@math.purdue.edu`