GRADIENT ESTIMATES OF HARMONIC FUNCTIONS AND TRANSITION DENSITIES FOR LÉVY PROCESSES

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Abstract. We prove gradient estimates for harmonic functions with respect to a d-dimensional unimodal pure-jump Lévy process under some mild assumptions on the density of its Lévy measure. These assumptions allow for a construction of an unimodal Lévy process in $\mathbb{R}^{d+2}$ with the same characteristic exponent as the original process. The relationship between the two processes provides a fruitful source of gradient estimates of transition densities. We also construct another process called a difference process which is very useful in the analysis of differential properties of harmonic functions. Our results extend the gradient estimates from [5] to a wide family of isotropic pure-jump process including a large class of subordinate Brownian motions.

1. Introduction

The main purpose of this paper is to investigate the growth properties of the gradient of functions which are harmonic with respect to some isotropic Lévy processes in $\mathbb{R}^d$. Another aim is to obtain gradient estimates of transition densities of these processes. Our main result concerning the gradient of harmonic functions is the following theorem.

Theorem 1.1. *Let X be an isotropic Lévy process in $\mathbb{R}^d$ satisfying assumptions (A) (formulated below). Let $D \subset \mathbb{R}^d$ be an open, nonempty set and let $f : \mathbb{R}^d \to [0, \infty)$ be a function which is harmonic with respect to X in D. Then $\nabla f(x)$ exists for any $x \in D$ and we have*

$$|\nabla f(x)| \le c \frac{f(x)}{\delta_D(x) \wedge 1}, \qquad x \in D, \tag{1}$$

where $\delta_D(x) = \mathrm{dist}(x, \partial D)$ and c is a constant depending only on the process X_t.

The proof of this result is based on a new observation about gradient of transition densities for Lévy processes (see Theorem 1.5) and a new concept of a difference process (see Section 4). It also uses recent results of P. Kim and A. Mimica [17] and K. Bogdan, T. Grzywny and M. Ryznar [2], [3], [14]. The dependence of the constant c in Theorem 1.1 on the process X will be further clarified in Remark 2.7.

Remark 1.2. We use a convention that for a radial function $f : \mathbb{R}^d \to \mathbb{R}$ we write $f(x) = f(r)$, if $x \in \mathbb{R}^d$ and $|x| = r$.

Assumptions (A).
(H0) $X = (X_t, t \ge 0)$ is a pure-jump isotropic Lévy process in $\mathbb{R}^d$ with the characteristic exponent ψ (i.e. $E^0 e^{i\xi X_t} = e^{-t\psi(\xi)}$). We assume that its Lévy measure is infinite and has the density $\nu(x) = \nu(|x|)$.

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(H1) $\nu(r)$ is nonincreasing, absolutely continuous such that $-\nu'(r)/r$ is nonincreasing, satisfies $\nu(r) \le a_1 \nu(r+1), r \ge 1$ and $\nu(r) \le a_1 \nu(2r), 0 < r \le 1$ for some constant a_1.

(H2) The scale invariant Harnack inequality holds for the process X (for the precise definition see Preliminaries).

The derivative $\nu'(r)$ is understood as a function (defined a.e. on $(0,\infty)$) such that $\nu(r) = -\int_r^\infty \nu'(\rho) d\rho$, $r > 0$. In fact, under the assumption that $-\nu'(r)/r$ is nonincreasing on the set where it is defined, we can always take a version which is well defined for each point $r > 0$ and $-\nu'(r)/r$ is nonincreasing on $(0,\infty)$. Throughout the whole paper we use that meaning of $\nu'(r)$. Note also that if $\nu(r)$ is convex then $-\nu'(r)/r$ is nonincreasing (in the above sense).

Observe that the condition (H2) is also necessary for the gradient estimate (1), since (1) implies the scale invariant Harnack inequality.

The next result exhibits some examples of classes of processes which satisfy assumptions (A). Before its formulation we introduce the definition of *a weak lower scaling condition* (cf. [2]). Let φ be a non-negative, non-zero function on $[0,\infty)$. We say that φ satisfies *a weak lower scaling condition* WLSC($\underline{\alpha}, \theta_0, \underline{C}$) if there are numbers $\underline{\alpha} > 0$, $\theta_0 \ge 0$ and $\underline{C} > 0$ such that

$$\varphi(\lambda\theta) \ge \underline{C} \lambda^{\underline{\alpha}} \varphi(\theta), \quad \text{for} \quad \lambda \ge 1,\, \theta \ge \theta_0.$$

Proposition 1.3. *Let us consider the following conditions:*

Assumptions (A1). *We assume (H0), (H1) and*

(H3) ψ satisfies WLSC($\underline{\alpha}, \theta_0, \underline{C}$).

Assumptions (A2). *We assume*

(H4) $X = (X_t, t \ge 0)$ is a subordinate Brownian motion, that is $X_t = B_{S_t}$, where $B = (B_t, t \ge 0)$ is the Brownian motion in $\mathbb{R}^d$ (with the generator Δ) and $S = (S_t, t \ge 0)$ is a subordinator independent of B. The Lévy measure of S is infinite.

(H5) The potential measure of S has a decreasing density.

(H6) The Lévy measure of S is infinite and has a decreasing density $\nu_S(r)$.

(H7) There exist constants $\delta \in (0,1]$, $\theta_0 > 0$, $\overline{C}$ such that the Laplace exponent ϕ of S satisfies

$$\frac{\phi'(\lambda\theta)}{\phi'(\theta)} \le \overline{C} \lambda^{-\delta}, \quad \textit{for} \quad \lambda \ge 1,\, \theta \ge \theta_0.$$

(H8) The density of the Lévy measure $\nu(x) = \nu(|x|)$ of the process X satisfies $\nu(r) \le a_1 \nu(r+1), r \ge 1$, for some constant $a_1 \ge 1$.

(H9) $d \ge 3$.

Assumptions (A3). *We assume (H4), (H7) and*

(H10) The Laplace exponent ϕ of S is a complete Bernstein function.

The Assumptions (A1) or (A2) or (A3) imply assumptions (A).

More concrete examples of processes satisfying assumptions (A) are in Section 7.

Remark 1.4. Conditions (H5), (H6), (H7), (H8) are exactly the same as conditions (A1), (A2), (A3), (1.2) in a recent, very interesting paper by P. Kim and A. Mimica [17]. Notation used in this paper and in [17] is slightly different.

Our gradient estimates of harmonic functions for Lévy processes are based on the following observation about the gradient of transition densities for these processes (cf. also Proposition 3.1).

Theorem 1.5. *Let X be a pure-jump isotropic Lévy process in $\mathbb{R}^d$ with the characteristic exponent ψ. We assume that its Lévy measure is infinite and has the density $\nu(x) = \nu(|x|)$ such that $\nu(r)$ is nonincreasing, absolutely continuous and $-\nu'(r)/r$ is nonincreasing. We denote transition densities of X by $p_t(x) = p_t(|x|)$. Then there exists a Lévy process $X_t^{(d+2)}$ in $\mathbb{R}^{d+2}$ with the characteristic exponent $\psi^{(d+2)}(\xi) = \psi(|\xi|)$, $\xi \in \mathbb{R}^{d+2}$ and the radial, radially nonincreasing transition density $p_t^{(d+2)}(x) = p_t^{(d+2)}(|x|)$ satisfying*

$$p_t^{(d+2)}(r) = \frac{-1}{2\pi r}\frac{d}{dr}p_t(r), \qquad r > 0. \tag{2}$$

Moreover, $p_t^{(d+2)}$ is continuous at any $x \neq 0$.

Remark 1.6. Note that if $X_t = B_{S_t}$ is a subordinate Brownian motion and the Levy measure of S is infinite then the above result is obvious and well-known. We note that the assumptions of Theorem 1.5 on $\nu(x)$ are automatically satisfied in this case.

Let φ be a nonnegative, nonzero function on $[0, \infty)$. We say that φ satisfies *a weak upper scaling condition* WUSC$(\overline{\alpha}, \theta_0, \overline{C})$ if there are numbers $\overline{\alpha} \in (0, 2)$, $\theta_0 \geq 0$ and $\overline{C} > 0$ such that

$$\varphi(\lambda\theta) \leq \overline{C}\lambda^{\overline{\alpha}}\varphi(\theta), \quad \text{for} \quad \lambda \geq 1,\ \theta \geq \theta_0.$$

Using Theorem 1.5 and the estimates of $p_t(x)$ from [2, Corollary 7, Theorem 21] we obtain the following result which seems to be of independent interest.

Corollary 1.7. *Let X be an isotropic pure-jump Lévy process in $\mathbb{R}^d$ satisfying assumptions (A). Then its transition density $p_t(x) = p_t(|x|)$ satisfies*

$$\left|\frac{d}{dr}p_t(r)\right| \leq c(d)\frac{1 \wedge t\psi^*(1/r)}{r^{d+1}}, \quad t, r > 0.$$

If additionally ψ satisfies WLSC($\underline{\alpha}, \theta_0, \underline{C}$), then

$$\left|\frac{d}{dr}p_t(r)\right| \leq c(d, \underline{\alpha})\frac{r}{\underline{C}^{(d+2)/\underline{\alpha}+1}}\left([\psi^-(1/t)]^{d+2} \wedge \frac{t\psi^*(1/r)}{r^{d+2}}\right), \quad t\psi^*(\theta_0) \leq 1/\pi^2.$$

If additionally ψ satisfies WLSC($\underline{\alpha}, \theta_0, \underline{C}$) and WUSC($\overline{\alpha}, \theta_0, \overline{C}$), then we have

$$\left|\frac{d}{dr}p_t(r)\right| \geq c^* r\left([\psi^-(1/t)]^{d+2} \wedge \frac{t\psi^*(1/r)}{r^{d+2}}\right), \quad t\psi^*(\theta_0/r_0) \leq 1, \quad r < r_0/\theta_0,$$

where $c^* = c^*(d, \underline{\alpha}, \overline{\alpha}, \underline{C}, \overline{C})$, $\quad r_0 = r_0(d, \underline{\alpha}, \overline{\alpha}, \underline{C}, \overline{C})$. Note that if the scaling conditions are global, that is $\theta_0 = 0$, then the last two estimates hold for all $t, r > 0$. Here ψ^- denotes the generalized inverse of $\psi^*(r) = sup_{\rho \leq r}\psi(\rho)$.

Theorem 1.1 implies the following result.

Corollary 1.8. *Let X be an isotropic Lévy process in $\mathbb{R}^d$ satisfying assumptions (A). Let $D \subset \mathbb{R}^d$ be an open, nonempty set, if X is not transient we assume additionally that D is bounded. Let $G_D(x, y)$ be the Green function of D corresponding* *to the process X for the set D (for the definition of G_D see Preliminaries). Then $\nabla_x G_D(x,y)$ exists for any $x, y \in D$, $x \neq y$, and we have*

$$|\nabla_x G_D(x,y)| \leq c \frac{G_D(x,y)}{\delta_D(x) \wedge |x-y| \wedge 1}, \qquad x \in D, \tag{3}$$

where $\delta_D(x) = \operatorname{dist}(x, \partial D)$.

The potential theory of Lévy processes, especially subordinate Brownian motions, has attracted a lot of attention during recent years see e.g. [6, 10, 9, 18, 19]. Our study is in the scope of this type of research. The assertion of Theorem 1.1 is well known for harmonic functions with respect to the Brownian motion and isotropic α-stable proceses [5], where explicit formulas for the Poisson kernel for a ball served as a main tool. For the processes treated in the present paper such formulas are not available and we had to take another approach based on Theorem 1.5. A result similar to Theorem 1.1 is also known for harmonic functions with respect to Schrödinger operators based on the Laplacian and the fractional Laplacian ([12], [5], [20]). Some probabilistic ideas used in our paper are, to some extent, similar to the concept of coupling from M. Cranston and Z. Zhao's paper [12]. The idea of coupling for Lévy processes was used by R. Schilling, P. Sztonyk and J. Wang in [28] where gradient estimates of the corresponding transition semigroups were derived. They are of the type $||\nabla P_t u||_\infty \leq c||u||_\infty f^{-1}(1/t)$, $u \in \mathcal{B}_b(\mathbb{R}^d)$, where $|\Re\psi(\xi)| \approx f(|\xi|)$ as $|\xi| \to \infty$. Recently, K. Kaleta and P. Sztonyk in [16] obtained also gradient estimates of transition densities for Lévy processes. Note that our sharp, two-sided estimates obtained in Corollary 1.7 are of different form than those obtained in [28] and [16]. The results in [28] and [16] are obtained under more general assumptions but they are not as sharp as ours.

Estimates of derivatives of harmonic functions with respect to some (not necessarily symmetric) α-stable processes were obtained by P. Sztonyk in [32]. However these estimates were obtained for a different class of processes and they are not pointwise (as our estimates) which is crucial in applications (see below). Moreover, P. Sztonyk in [32] obtained gradient estimates only for $\alpha \geq 1$.

It seems that for applications the most important are Corollaries 1.7 and 1.8. For example they could be used to study operators $L + b\nabla$ (where L is the generator of a Lévy process in $\mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}$). Such operators (when $L = -(-\Delta)^{\alpha/2}$) are intensively studied both in the theory of partial differential equations (see e.g. [30]) and in the theory of stochastic processes (see e.g. [4], [8]). Especially the techniques used by K. Bogdan, T. Jakubowski in [4] and by Z.-Q. Chen, P. Kim, R. Song in [8] demand pointwise gradient estimates of transition densities and a Green function exactly of the type presented in our paper. It seems that our estimates would allow to extend results from [4] and [8] (obtained there for the fractional Laplacian) to more general generators of Lévy processes. Our estimates also seem to be useful in the study of the spectral theory related to Lévy processes and Schrödinger operators based on their generators.

The paper is organized as follows. Section 2 is preliminary. In Section 3 we prove Theorem 1.5. Section 4 concerns the difference process. In Section 5 we prove some auxiliary facts concerning the Green function and the Lévy measure. In Section 6 we prove Theorem 1.1. In the last section we present examples of processes satisfying assumptions (A) but we also provide an example of an unimodal isotropic pure-jump

Lévy process and a harmonic function with respect to X for which the gradient does not exist at some point.

2. Preliminaries

For $x \in \mathbb{R}^d$ and $r > 0$ we let $B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r\}$. By $a \wedge b$ we denote $\min(a,b)$ for $a, b \in \mathbb{R}$. When $D \subset \mathbb{R}^d$ is an open set we denote by $\mathcal{B}(D)$ a family of Borel subsets of D.

A Borel measure on $\mathbb{R}^d$ is called *isotropic unimodal* if on $\mathbb{R}^d \setminus \{0\}$ it is absolutely continuous with respect to the Lebesgue measure and has a finite radial, radially nonincreasing density function (such measures may have an atom at the origin).

A Lévy process $X = (X_t, t \geq 0)$ in $\mathbb{R}^d$ is called isotropic unimodal if its transition probability $p_t(dx)$ is isotropic unimodal for all $t > 0$. When additionally X is a pure-jump process then the following Lévy-Khintchine formula holds for $t > 0$ and $\xi \in \mathbb{R}^d$,

$$E^0 e^{i\xi X_t} = \int_{\mathbb{R}^d} e^{i\xi x} p_t(dx) = e^{-t\psi(\xi)} \quad \text{where} \quad \psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi x))\nu(dx).$$

ψ is the characteristic exponent of X and ν is the Lévy measure of X. E^0 is the expected value for the process X starting from 0. Recall that a Lévy measure is a measure concentrated on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$. Isotropic unimodal pure-jump Lévy measures are characterized in [33] by unimodal Lévy measures $\nu(dx) = \nu(x)\, dx = \nu(|x|)\, dx$.

Unless explicitly stated otherwise in what follows we assume that X is a pure-jump isotropic unimodal Lévy process in $\mathbb{R}^d$ with (isotropic unimodal) infinite Lévy measure ν. Then for any $t > 0$ the measure $p_t(dx)$ has a radial, radially nonincreasing density function $p_t(x) = p_t(|x|)$ on $\mathbb{R}^d$ with no atom at the origin. However, it may happen that $p_t(0) = \infty$, for some $t > 0$. As usual, we denote by P^x and E^x the probability measure and the corresponding expectation for the the process starting from $x \in \mathbb{R}^d$.

The process X is said to be transient if $P^0(\lim_{t\to\infty} |X_t| = \infty) = 1$. For $d \geq 3$ the process X is always transient (see e.g. [14], the remark after Lemma 5).

For a transient process by U we denote the potential kernel for the process X. That is

$$U(x) = \int_0^\infty p_t(x)\, dt, \quad x \in \mathbb{R}^d.$$

By $U^{(d+2)}$ we denote the potential kernel for the process $X^{(d+2)}$ defined in Theorem 1.5. Since the process $X^{(d+2)}$ lives in at least three-dimensional space then $U^{(d+2)}(x) < \infty$, $x \neq 0$. T. Grzywny in [14] obtained estimates of the potential kernel in terms of the symbol ψ, which play an important role in the present paper.

We define the maximal characteristic function $\psi^*(r) = \sup_{s \leq r} \psi(s)$, where $r \geq 0$. We have [2, Proposition 2] $\psi(r) \leq \psi^*(r) \leq \pi^2 \psi(r)$, $r \geq 0$. The function ψ^* has the property [14, Lemma 1],

$$\psi^*(r) \leq 2\frac{1+s^2}{s^2}\psi^*(sr), \quad r, s > 0. \tag{4}$$

In the sequel the following nondecreasing function will play an important role in our development

$$L(r) = \left(\psi^*\left(\frac{1}{r}\right)\right)^{-1/2}, \quad r > 0,$$

and $L(0) = 0$. As an immediate consequence of (4) we have

$$L(sr) \le \sqrt{2(1+s^2)}L(r), \quad r, s > 0. \tag{5}$$

This property will be frequently used throughout the paper without further mention while comparing values of L at points with fixed ratio. There are many important quantities related to the process X, which enjoy precise estimates in terms of $L(r)$. We have [2, Corollary 7],

$$p_t(x) \le \frac{ct}{L^2(|x|)|x|^d}, \quad t > 0,\, x \in \mathbb{R}^d, \tag{6}$$

$$\nu(x) \le \frac{c}{L^2(|x|)|x|^d}, \quad x \in \mathbb{R}^d, \tag{7}$$

where $c = c(d)$. Under some further conditions (6)-(7) can be reveresed ([2]). Note that the upper bound of the Lévy density yields

$$\limsup_{r \searrow 0} r^{d+2}\nu(r) \le \limsup_{r \searrow 0} \frac{cr^2}{L^2(r)} = 0, \tag{8}$$

and

$$\limsup_{r \to \infty} r^{d}\nu(r) \le \limsup_{r \to \infty} \frac{c}{L^2(r)} = 0. \tag{9}$$

For the proof that $\limsup_{r\searrow 0} \frac{r^2}{L^2(r)} = 0$, see [3, Lemma 2.5].

The first exit time of an open, nonempty set $D \subset \mathbb{R}^d$ of the process X is defined by $\tau_D = \inf\{t > 0 : \, X_t \notin D\}$.

Definition 2.1. A Borel function $f : \mathbb{R}^d \to \mathbb{R}$ is called *harmonic* with respect to the process X in an open, nonempty set $D \subset \mathbb{R}^d$ if for any bounded, open, nonempty set B, such that $\overline{B} \subset D$

$$f(x) = E^x\left(f\left(X(\tau_B)\right)\right), \qquad x \in B.$$

We understand that the expectation is absolutely convergent.

Definition 2.2. The *scale invariant Harnack inequality* holds for the process X if there exists a constant a_2 such that for any $x_0 \in R^d$, $r \in (0, 1]$, and any function h nonnegative on $\mathbb{R}^d$ and harmonic in a ball $B(x_0, r)$,

$$\sup_{x \in B(x_0, r/2)} h(x) \le a_2 \inf_{x \in B(x_0, r/2)} h(x).$$

Let $D \subset \mathbb{R}^d$ be an open, nonempty set. We define a *killed process* X_t^D by $X_t^D = X_t$ if $t < \tau_D$ and $X_t^D = \partial$ otherwise, where ∂ is some point adjoined to D (usually called cemetary). The transition density for X_t^D on D is given by

$$p_D(t, x, y) = p_t(x - y) - E^x(p_{t-\tau_D}(X(\tau_D), y),\, t > \tau_D), \quad x, y \in D,\, t > 0,$$

that is for any Borel set $A \subset \mathbb{R}^d$ we have

$$P^x(X_t^D \in A) = \int_A p_D(t, x, y)\, dy, \quad x \in D,\, t > 0.$$

We have $p_D(t,x,y) = p_D(t,y,x)$, $x,y \in D$, $t > 0$. We define the *Green function* for X_t^D by

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)\,dt, \quad x,y \in D,$$

$G_D(x,y) = 0$ if $x \notin D$ or $y \notin D$. For any Borel set $A \subset \mathbb{R}^d$ we have

$$E^x \int_0^{\tau_D} 1_A(X_t)\,dt = \int_A G_D(x,y)\,dy, \quad x \in D.$$

In particular if we set $A = D$ we obtain

$$E^x \tau_D = \int_D G_D(x,y)\,dy, \quad x \in D.$$

We have $G_D(x,y) = G_D(y,x)$, $x,y \in D$. For a fixed $y \in D$ the function $x \to G_D(x,y)$ is harmonic with respect to X in $D \setminus \{y\}$. The estimates of $E^x\tau_D$ when D is a ball play an important role in the paper. Here we record very useful upper and lower bounds in terms of the function L (see e.g. [3, Lemmas 2.3, 2.7], see also [27]).

Lemma 2.3. *There exist an absolute constant C_1, and a constant $C_2 = C_2(d)$ such that for any $r > 0$ we have*

$$E^x \tau_{B(0,r)} \le C_1 L(r) L(\delta(x)), \quad x \in B(0,r)$$

and

$$E^x \tau_{B(0,r)} \ge C_2 L^2(r), \quad x \in B(0,r/2),$$

where $\delta(x) = \delta_{B(0,r)}(x)$.

Let $D \subset \mathbb{R}^d$ be an open, nonempty set. The distribution $P^x(X(\tau_D) \in \cdot)$ is called the *harmonic measure* with respect to X. The harmonic measure for Borel sets $A \subset (\overline{D})^c$ is given by the Ikeda-Watanabe formula [15],

$$P^x(X(\tau_D) \in A) = \int_A \int_D G_D(x,y)\nu(y-z)\,dy\,dz, \quad x \in D. \tag{10}$$

When $D \subset \mathbb{R}^d$ is a bounded, open Lipschitz set then we have [31], [23],

$$P^x(X(\tau_D) \in \partial D) = 0, \quad x \in D. \tag{11}$$

It follows that for such sets D the Ikeda-Watanabe formula (10) holds for any Borel set $A \subset D^c$. Let $D \subset \mathbb{R}^d$ be an open, nonempty set. For any $s > 0$, $x \in D$, $z \in (\overline{D})^c$ put

$$h_D(x,s,z) = \int_D p_D(s,x,y)\nu(y-z)\,dy. \tag{12}$$

By Ikeda-Watanabe formula [15] for any Borel $A \subset (0,\infty)$, $B \subset (\bar{D})^c$ we have

$$P^x(\tau_D \in A, X(\tau_D) \in B) = \int_A \int_B h_D(x,s,z)\,dz\,ds, \quad x \in D. \tag{13}$$

If (11) holds then we can take $B \subset D^c$ in (13).

From [3, Lemma 2.1] we have the following estimate.

Lemma 2.4. *Let $z \in \mathbb{R}^d$, $s > 0$, $D \subset B(z,s)$ be a bounded, open, nonempty set and $y \in D \cap B(z,s/2)$. There is a constant $c = c(d)$ such that*

$$P^y(X(\tau_D) \in B^c(z,s)) \le c\frac{E^y(\tau_D)}{L^2(s)}.$$

Important examples of isotropic unimodal Lévy processes are subordinate Brownian motions. By $S = (S_t, t \geq 0)$ we denote a *subordinator* i.e. a nondecreasing Lévy process starting from 0. The Laplace transform of S is of the form

$$Ee^{-\lambda S_t} = e^{-t\phi(\lambda)}, \quad \lambda \geq 0,\, t \geq 0,$$

where ϕ is called the Laplace exponent of S. ϕ is a Bernstein function and has the following representation

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda u})\, \nu_S(du) \tag{14}$$

where $b \geq 0$ and ν_S is a Lévy measure on $(0,\infty)$ such that $\int_{(0,\infty)} (1 \wedge u)\, \nu_S(du) < \infty$.

Let $B = (B_t, t \geq 0)$ be a Brownian motion in $\mathbb{R}^d$ (with a generator Δ) and let S be an independent subordinator. We define a new process $X_t = B_{S_t}$ and call it a *subordinate Brownian motion.* Let us assume that $b = 0$ and $\nu_S(0,\infty) = \infty$ in (14). Then X is a Lévy process with the characteristic exponent $\psi(\xi) = \phi(|\xi|^2)$ and the Lévy measure $\nu(dx) = \nu(x)\, dx = \nu(|x|)\, dx$ given by [26, Theorem 30.1],

$$\nu(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} \exp\left(-\frac{r^2}{4t}\right) \nu_S(dt), \quad r > 0.$$

The next lemma seems to be known but we could not find any reference so we decided to present its short proof.

Lemma 2.5. *Let X be a pure-jump isotropic Lévy process in $\mathbb{R}^d$. We assume that its Lévy measure is infinite and has the density $\nu(x) = \nu(|x|)$ which is radially nonincreasing. Then for each $t > 0$ the density function $p_t(x)$ of the process is continuous on $\mathbb{R}^d \setminus \{0\}$.*

Proof. Let p_t be the distribution of X_t. It is well known that under above assumptions for each $t > 0$ the measure p_t has a radial, nonincreasing density function $p_t(x)$ on $\mathbb{R}^d \setminus \{0\}$ and p_t has no atom at $\{0\}$.

Let us denote $f_t(x) = \int_{\mathbb{R}^d} p_{t/2}(x-y) p_{t/2}(y)\, dy$, $t > 0$, $x \in \mathbb{R}^d$. We have $p_t(x) = f_t(x)$ a.s. so it is enough to show that for each $t > 0$ the function f_t is continuous on $\mathbb{R}^d \setminus \{0\}$.

Fix $t > 0$, $z \in \mathbb{R}^d$, $z \neq 0$ and $\varepsilon > 0$. Let $M = \sup_{y \in B^c(0,|z|/2)} p_{t/2}(y)$. Take $\delta \in (0, |z|/4)$ such that $\int_{B(0,\delta)} p_{t/2}(y)\, dy < \varepsilon/(4M)$. For any $x \in B(z, |z|/4)$ we have

$$\int_{B(0,\delta)} p_{t/2}(x-y) p_{t/2}(y)\, dy \leq M \int_{B(0,\delta)} p_{t/2}(y)\, dy < \varepsilon/4. \tag{15}$$

Denote $f^{(1)}(x) = \int_{B(0,\delta)} p_{t/2}(x-y) p_{t/2}(y)\, dy$ and $f^{(2)}(x) = \int_{B^c(0,\delta)} p_{t/2}(x-y) p_{t/2}(y)\, dy$. (15) implies that for $x \in B(z, |z|/4)$ we have $|f^{(1)}(x) - f^{(1)}(z)| \leq \varepsilon/2$. On the other hand note that $f^{(2)}(x)$ is the convolution of the function $p_{t/2}(y) \in L^1(\mathbb{R}^d)$ and the bounded function $1_{[\delta,\infty)}(|y|) p_{t/2}(y)$. Hence $f^{(2)}(x)$ is continuous on $\mathbb{R}^d$. It follows that $f_t(x) = f^{(1)}(x) + f^{(2)}(x)$ is continuous at $x = z$. □

Lemma 2.6. *For any $\varepsilon \in (0,1]$ let $f_\varepsilon \in L^1(\mathbb{R}^d)$ and let $f \in L^1(\mathbb{R}^d)$. Assume that all f_ε, f are nonnegative, continuous, radial, radially nonincreasing and $f_\varepsilon \to f$ weakly as $\varepsilon \to 0$ (as measures on $\mathbb{R}^d$). Then the convergence is pointwise at any $x \neq 0$.*

Proof. Let $0 < a < b < \infty$. From the weak convergence

$$\lim_{\varepsilon \to 0} \int_a^b f_\varepsilon(r) r^{d-1}\, dr = \int_a^b f(r) r^{d-1}\, dr.$$

By monotonicity $\int_a^b f(r) r^{d-1}\, dr \le f(a) b^{d-1}(b-a)$ and $f_\varepsilon(b) a^{d-1}(b-a) \le \int_a^b f_\varepsilon(r) r^{d-1}\, dr$. It follows that

$$\limsup_{\varepsilon \to 0} f_\varepsilon(b) a^{d-1} \le f(a) b^{d-1}.$$

Using continuity of f and passing $a \nearrow b$ we obtain $\limsup_{\varepsilon \to 0} f_\varepsilon(b) \le f(b)$. By a symmetric argument we have $\liminf_{\varepsilon \to 0} f_\varepsilon(a) \ge f(a)$. □

Now we will show Proposition 1.3.

Proof of Proposition 1.3. First, we show that assumptions (A1) imply (A). If $d \ge 3$ then [14, Theorem 1] gives (H2). If $d \le 2$ then Theorem 1.5 and [14, Corollary 6] give (H2).

In the next step we prove that assumptions (A2) imply (A). Recall that conditions (H5), (H6), (H7), (H8) are exactly the same as conditions (A1), (A2), (A3), (1.2) in [17]. (H2) follows from [17, Theorem 1.2]. Remark 1.6 implies that $\nu(r)$ is nonincreasing, absolutely continuous and $-\nu'(r)/r$ is nonincreasing. Now we will show that $\nu(r) \le a\nu(2r), r \in (0, 1]$ for some constant a. It is clear that the scaling property for ϕ' (H7) implies that $\phi'(r^{-2}) \le a'\phi'((2r)^{-2})$, $r \in (0, 1/\sqrt{4\theta_0}]$. Since $\nu(r) \approx \phi'(r^{-2}) r^{-d-2}, r \in (0, 1]$ (see [17, Proposition 4.2])) we obtain there is a constant a'' such that $\nu(r) \le a''\nu(2r), r \in (0, 1/\sqrt{4\theta_0}]$. Clearly this inequality holds for all $r \in (0, 1]$ with (possibly) a different constant.

Finally, we justify that assumptions (A3) imply (A). This again follows from arguments presented in the paper by P. Kim and A. Mimica [17]. (H4) and (H10) imply (H5) and (H6). (H4) and (H10) imply also (H8), see [17, Remark 4.3]. So (H4), (H5), (H6), (H7), (H8) hold. Hence we can use [17, Theorem 1.2] and get (H2). Remark 1.6 implies that $\nu(r)$ is nonincreasing, absolutely continuous and $-\nu'(r)/r$ is nonincreasing. The fact that $\nu(r) \le a\nu(2r), r \in (0, 1]$ for some constant a can be shown in the same way as in case (A2). □

Remark 2.7. All constants appearing in this paper are positive and finite. We write $\kappa = \kappa(a, \dots, z)$ to emphasize that κ depends only on $a, \dots, z$. We adopt the convention that constants denoted by c (or c_1, c_2) may change their value from one use to the next. In the whole paper, unless is explicitly stated otherwise, we understand that constants denoted by c (or c_1, c_2) depend on d, a_1, a_2, where a_1, a_2 appear in (H1) and Definition 2.2, respectively. In particular, it applies to the constant c in (1).

3. The derivative of the transition density

We denote the Fourier transform of $f \in L^1(\mathbb{R}^d)$ by $\mathcal{F}f(y) = \int_{\mathbb{R}^d} e^{-ixy} f(x)\, dx$, $y \in \mathbb{R}^d$ and the inverse Fourier transform of $f \in L^1(\mathbb{R}^d)$ by $\tilde{\mathcal{F}}f(y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixy} f(x)\, dx$, $y \in \mathbb{R}^d$. It is well known that for any real, radial $f \in L^1(\mathbb{R}^d)$ we have $\mathcal{F}f(y) = \mathcal{F}^{(d)} f(|y|)$, $y \in \mathbb{R}^d$, $y \ne 0$, where

$$\mathcal{F}^{(d)} f(R) = (2\pi)^{d/2} \int_0^\infty \frac{J_{\frac{d-2}{2}}(rR)}{(rR)^{\frac{d-2}{2}}} f(r) r^{d-1}\, dr, \quad R > 0.$$

Here J_α is the Bessel function of order α. Similarly for any real, radial $f \in L^1(\mathbb{R}^d)$ we have $\tilde{\mathfrak{F}} f(y) = \tilde{\mathfrak{F}}^{(d)} f(|y|)$, $y \in \mathbb{R}^d$, $y \neq 0$, where $\tilde{\mathfrak{F}}^{(d)} f(R) = (2\pi)^{-d} \mathfrak{F}^{(d)} f(R)$, $R > 0$.

We will use the following result from [13]. Let $f : [0, \infty) \to \mathbb{R}$ be a Borel function satisfying $\int_0^\infty |f(r)|(r^{d-1} + r^{d+1}) dr < \infty$. Then we have

$$\frac{d}{dR} (\tilde{\mathfrak{F}}^{(d)} f)(R) = -2\pi R \tilde{\mathfrak{F}}^{(d+2)} f(R), \quad R > 0. \tag{16}$$

We first prove Proposition 3.1 which is a version of Theorem 1.5 with slightly changed assumptions. We will use this proposition in the proof of Theorem 1.5 but it seems that Proposition 3.1 is of independent interest.

Proposition 3.1. *Let X be an isotropic Lévy process in $\mathbb{R}^d$ with the characteristic exponent ψ and the transition density $p_t(x) = p_t(|x|)$. We assume that its Lévy measure has the density $\nu(x) = \nu(|x|)$. We further assume that ψ satisfies $\lim_{\rho\to\infty}(\psi(\rho)/\log(\rho)) = \infty$ and $\nu(r)$ is nonincreasing. Then there exists a Lévy process $X^{(d+2)}$ in $\mathbb{R}^{d+2}$ with the characteristic exponent $\psi^{(d+2)}(\xi) = \psi(|\xi|)$, $\xi \in \mathbb{R}^{d+2}$ and the transition density $p_t^{(d+2)}(x) = p_t^{(d+2)}(|x|)$ satisfying*

$$p_t^{(d+2)}(r) = \frac{-1}{2\pi r} \frac{d}{dr} p_t(r), \qquad r > 0.$$

Proof. Put $s_t(r) := e^{-t\psi(r)}$, $r \geq 0$. By the fact that ψ satisfies $\lim_{\rho\to\infty}(\psi(\rho)/\log(\rho)) = \infty$ we obtain that $\int_0^\infty |s_t(r)|(r^{d-1} + r^{d+1}) dr < \infty$, for any $t > 0$. We have $p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ixy} e^{-t\psi(y)} \, dy$, $x \in \mathbb{R}^d$, so $p_t(R) = \tilde{\mathfrak{F}}^{(d)} s_t(R)$, $R > 0$.

Now let us define

$$p_t^{(d+2)}(x) = (2\pi)^{-d-2} \int_{\mathbb{R}^{d+2}} e^{ixy} e^{-t\psi(y)} \, dy, \quad x \in \mathbb{R}^{d+2}, \, t > 0.$$

By (16) we have

$$p_t^{(d+2)}(R) = \tilde{\mathfrak{F}}^{(d+2)} s_t(R) = \frac{-1}{2\pi R} \frac{d}{dR} (\tilde{\mathfrak{F}}^{(d)} s_t)(R) = \frac{-1}{2\pi R} \frac{d}{dR} p_t(R).$$

Note that $p_t(R)$ is nonincreasing so $p_t^{(d+2)}(R) \geq 0$, $R > 0$.

It follows that $\mathfrak{F} p_t^{(d+2)}(x) = \mathfrak{F}\tilde{\mathfrak{F}}(e^{-t\psi(\cdot)})(x) = e^{-t\psi(x)}$, $x \in \mathbb{R}^{d+2}$, $t > 0$. Since $\psi(0) = 0$ we observe that $p_t^{(d+2)}(x)$ is a probability density. Moreover, this implies that $p_t^{(d+2)} * p_s^{(d+2)} = p_{t+s}^{(d+2)}$, $s, t > 0$ and $p_t^{(d+2)}$ tends weakly to δ_0 as $t \to 0$, where δ_0 is the Dirac delta at 0.

In consequence there exists a Lévy process $\{X_t^{(d+2)}\}_{t\geq 0}$ in $\mathbb{R}^{d+2}$ with the transition density $p_t^{(d+2)}(x)$ and the Lévy-Khinchin exponent ψ.

□

Proof of Theorem 1.5. Let us define

$$\nu^{(d+2)}(R) = -\frac{1}{2\pi R} \frac{d\nu}{dR}(R), \quad R > 0,$$

and $\nu^{(d+2)}(x) = \nu^{(d+2)}(|x|)$, $x \in \mathbb{R}^{d+2}$, $x \neq 0$. Let $\nu^{(d+2)}$ be the measure on $\mathbb{R}^{d+2}$ given by $\nu^{(d+2)}(\{0\}) = 0$ and $\nu^{(d+2)}(dx) = \nu^{(d+2)}(x)dx$, $x \in \mathbb{R}^{d+2}$, $x \neq 0$.

Now we will show that $\int_{R^{d+2}} (1 \wedge |x|^2) \, \nu^{(d+2)}(dx) < \infty$. Clearly, $(-\nu)'(|x|)(2\pi|x|)^{-1} \geq 0$. Note that we have to show $\int_{\mathbb{R}^{d+2}} (-\nu)'(|x|)(2\pi|x|)^{-1}(1 \wedge |x|^2) \, dx < \infty$. It is enough to prove $\int_0^1 (-\nu)'(r) r^{d+2}\, dr < \infty$ and $\int_1^\infty (-\nu)'(r) r^d\, dr < \infty$. Integrating by parts we get

$$\int_\varepsilon^1 (-\nu)'(r) r^{d+2}\, dr = -\nu(r) r^{d+2}|_\varepsilon^1 + (d+2)\int_\varepsilon^1 \nu(r) r^{d+1}\, dr,$$

where $\varepsilon \in (0,1)$ is arbitrary. Since $\int_0^1 \nu(r) r^{d+1}\, dr < \infty$ we must have $\liminf_{r\to 0} \nu(r) r^{d+2} = 0$. It follows that $\int_0^1 (-\nu)'(r) r^{d+2}\, dr < \infty$. Again integrating by parts we obtain

$$\int_1^N (-\nu)'(r) r^d\, dr = -\nu(r) r^d|_1^N + d\int_1^N \nu(r) r^{d-1}\, dr,$$

where $N > 1$ is arbitrary. It follows that $\int_1^\infty (-\nu)'(r) r^d\, dr < \infty$.

Hence the measure $\nu^{(d+2)}$ satisfies the conditions of a Lévy measure in $\mathbb{R}^{d+2}$. Let $X^{(d+2)} = (X_t^{(d+2)}, t \geq 0)$ be a pure-jump Levy process in $\mathbb{R}^{d+2}$ with a Lévy measure $\nu^{(d+2)}$. One can easily check that $\nu^{(d+2)}(\mathbb{R}^{d+2}) = \infty$. Indeed,

$$\int_\epsilon^1 (-\nu)'(r) r^d\, dr = -\nu(r) r^d|_\epsilon^1 + d\int_\epsilon^1 \nu(r) r^{d-1}\, dr \to \infty, \quad \epsilon \searrow 0.$$

Note that the Levy measure $\nu^{(d+2)}$ has the density which is radial and radially nonincreasing. Let $p_t^{(d+2)}$ be the distribution of $X_t^{(d+2)}$. It follows that for each $t > 0$ the measure $p_t^{(d+2)}$ has a radial, nonincreasing bounded density function $p_t^{(d+2)}(x) = p_t^{(d+2)}(|x|)$ on $\mathbb{R}^{d+2}$, which is continuous by Lemma 2.5.

Let $\psi^{(d+2)}$ be the characteristic exponent for the process $X^{(d+2)}$. Now our aim is to show that $\psi^{(d+2)}(R) = \psi(R)$, $R > 0$. We have

$$\psi^{(d+2)}(\xi) = \int_{\mathbb{R}^{d+2}} (1 - \cos(\xi x))\, \nu^{(d+2)}(x)\, dx, \quad \xi \in \mathbb{R}^{d+2}.$$

So to prove $\psi^{(d+2)}(R) = \psi(R)$, $R > 0$ it is enough to show that

$$\psi(\xi) = \int_{\mathbb{R}^{d+2}} (1 - \cos(\xi x)) \left(-\frac{1}{2\pi|x|} (\nu)'(|x|) \right) dx, \quad \xi \in \mathbb{R}^{d+2}.$$

Hence it is sufficient to prove for $R > 0$,

$$\psi(R) = \int_0^\infty \left(\omega_{d+1} - (2\pi)^{\frac{d+2}{2}} (rR)^{-\frac{d}{2}} J_{\frac{d}{2}}(rR) \right) \left(-\frac{1}{2\pi r} \frac{d\nu}{dr}(r) \right) r^{d+1}\, dr,$$

where $\omega_d = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$. Since ν is the density of the Lévy measure of X in $\mathbb{R}^d$ we have

$$\begin{aligned}
\psi(R) &= \int_0^\infty \left(\omega_{d-1} - (2\pi)^{\frac{d}{2}} (rR)^{-(\frac{d-2}{2})} J_{\frac{d-2}{2}}(rR) \right) \nu(r) r^{d-1}\, dr \\
&= \int_0^\infty \left(\omega_{d-1} r^{d-1} - (2\pi)^{\frac{d}{2}} (rR)^{\frac{d}{2}} J_{\frac{d-2}{2}}(rR) R^{1-d} \right) \nu(r)\, dr.
\end{aligned}$$

Using the property of Bessel functions $\frac{d}{ds}(s^\alpha J_\alpha(s)) = s^\alpha J_{\alpha-1}(s)$ ($\alpha \in (-1/2, \infty)$, $s > 0$) this is equal to

$$I = \int_0^\infty \frac{d}{dr} \left(\frac{\omega_{d-1}}{d} r^d - (2\pi)^{\frac{d}{2}} (rR)^{\frac{d}{2}} J_{\frac{d}{2}}(rR) R^{-d} \right) \nu(r)\, dr.$$

By asymptotics of the Bessel function $J_{\frac{d}{2}}(r)$ at zero we show that

$$\frac{\omega_{d-1}}{d} r^d - (2\pi)^{\frac{d}{2}} (rR)^{\frac{d}{2}} J_{\frac{d}{2}}(rR) R^{-d} \approx C r^{d+2} R^{2+d/2}.$$

Hence, applying (8),

$$\lim_{r\to 0}\left(\frac{\omega_{d-1}}{d}r^d-(2\pi)^{\frac d2}(rR)^{\frac d2}J_{\frac d2}(rR)R^{-d}\right)\nu(r)=\lim_{r\to 0}r^{d+2}\nu(r)=0.$$

Using the fact the Bessel function $J_{\frac d2}(r)$ is bounded at ∞, we show, applying (9), that

$$\lim_{r\to\infty}\left|\frac{\omega_{d-1}}{d}r^d-(2\pi)^{\frac d2}(rR)^{\frac d2}J_{\frac d2}(rR)R^{-d}\right|\nu(r)=\lim_{r\to\infty}r^{d}\nu(r)=0.$$

This justifies that by integrating by parts we obtain

$$I=\int_0^\infty\left(\omega_{d+1}-(2\pi)^{\frac{d+2}{2}}(rR)^{-\frac d2}J_{\frac d2}(rR)\right)\left(-\frac{1}{2\pi r}\frac{d\nu}{dr}(r)\right)r^{d+1}\,dr.$$

So we have finally shown that $X^{(d+2)}$ has the characteristic exponent ψ.

Our next aim is to show (2). For any $\varepsilon\in(0,1]$ let $X_{(\varepsilon)}=(X_{(\varepsilon),t},t\geq 0)$ be the Lévy process in $\mathbb{R}^d$ with the characteristic exponent $\psi_\varepsilon(\xi)=\psi(\xi)+\varepsilon|\xi|$, $\xi\in\mathbb{R}^d$. Let $p_{\varepsilon,t}$ be the distribution of $X_{(\varepsilon),t}$. It follows that for each $t>0$ the measure $p_{\varepsilon,t}$ has a radial, nonincreasing bounded density function $p_{\varepsilon,t}(x)=p_{\varepsilon,t}(|x|)$ on $\mathbb{R}^d$. For any $t>0$ clearly, $p_{\varepsilon,t}\to p_t$ weakly as $\varepsilon\searrow 0$. By Lemma 2.5 all densities $p_{\varepsilon,t}(x)$, $p_t(x)$ are continuous on $\mathbb{R}^d\setminus\{0\}$. Hence by Lemma 2.6 for any $t>0$, $x\in\mathbb{R}^d$, $x\neq 0$ we have $p_{\varepsilon,t}(x)\to p_t(x)$ as $\varepsilon\searrow 0$.

By Proposition 3.1 there exists a Lévy process $X^{(d+2)}_{(\varepsilon)}=(X^{(d+2)}_{(\varepsilon),t},t\geq 0)$ in $\mathbb{R}^{d+2}$ with the characteristic exponent $\psi^{(d+2)}_\varepsilon(\xi)=\psi(|\xi|)+\varepsilon|\xi|$, $\xi\in\mathbb{R}^{d+2}$. Let $p^{(d+2)}_{\varepsilon,t}$ be the distribution of $X^{(d+2)}_{(\varepsilon),t}$. It follows that for each $t>0$ the measure $p^{(d+2)}_{\varepsilon,t}$ has a radial, nonincreasing bounded density function $p^{(d+2)}_{\varepsilon,t}(x)=p^{(d+2)}_{\varepsilon,t}(|x|)$ on $\mathbb{R}^{d+2}$. For any $t>0$ clearly, $p^{(d+2)}_{\varepsilon,t}\to p^{(d+2)}_t$ weakly as $\varepsilon\searrow 0$. All densities $p^{(d+2)}_{\varepsilon,t}(x)$, $p^{(d+2)}_t(x)$ are continuous on $\mathbb{R}^{d+2}\setminus\{0\}$.

Fix $0<r_1<r_2<\infty$. By Proposition 3.1 we have

$$p_{\varepsilon,t}(r_2)-p_{\varepsilon,t}(r_1)=\int_{r_1}^{r_2}\frac{\partial}{\partial r}p_{\varepsilon,t}(r)\,dr=\int_{r_1}^{r_2}\frac{-1}{2\pi r}p^{(d+2)}_{\varepsilon,t}(r)\,dr.$$

Since for any $r>0$ we have $p_{\varepsilon,t}(r)\to p_t(r)$ as $\varepsilon\searrow 0$ and $p^{(d+2)}_{\varepsilon,t}\to p^{(d+2)}_t$ weakly as $\varepsilon\searrow 0$ we obtain

$$p_t(r_2)-p_t(r_1)=\int_{r_1}^{r_2}\frac{-1}{2\pi r}p^{(d+2)}_t(r)\,dr.$$

By continuity of $p^{(d+2)}_t(r)$ we arrive at (2).

□

4. The difference process

Let X be a pure-jump isotropic unimodal Lévy process in $\mathbb{R}^d$ with an infinite Lévy measure $\nu(dx)=\nu(x)\,dx$. The process has the transition density $p_t(x)$, which as a function of x is also radially nonincreasing. We will use the following notation $\hat{x}=(-x_1,x_2,\dots,x_d)$ for $x=(x_1,x_2,\dots,x_d)$, $D_+=\{(x_1,x_2,\dots,x_d)\in D:\ x_1>0\}$, $D_-=\{(x_1,x_2,\dots,x_d)\in D:\ x_1<0\}$ for $D\subset\mathbb{R}^d$.

The purpose of this section is to construct a Markov process $\tilde{X}_t$ on $\mathbb{R}^d_+=\{x\in\mathbb{R}^d: x_1>0\}$ with a sub-Markov transition density $p_t(x-y)-p_t(\hat{x}-y)$ and derive its basic properties. We call $\tilde{X}_t$ *the difference process*. To provide some intuitions we briefly present its construction when X is a subordinate Brownian motion. In such case the construction is easy and intuitive. Then we present the construction in the general case.

Let $X_t = B_{S_t}$, where B_t is the Brownian motion in $\mathbb{R}^d$ (with the generator Δ) and S_t is a subordinator independent of B_t having infinite Lévy measure. Denote by $g_t(x)$ the transition density of B_t. The transition density of X_t is given by $p_t(x) = \int_0^\infty g_s(x) P(S_t \in ds)$. Let $\tau^B_{\mathbb{R}^d_+} = \inf\{t \geq 0 : B_t \notin \mathbb{R}^d_+\}$ and $\tilde{B}_t$ be the Brownian motion killed on exiting $\mathbb{R}^d_+$ that is

$$\tilde{B}_t = \begin{cases} B_t, & \text{for} \quad t < \tau^B_{\mathbb{R}^d_+}, \\ \partial, & \text{for} \quad t \geq \tau^B_{\mathbb{R}^d_+}. \end{cases}$$

Here we augment $\mathbb{R}^d_+$ by an extra point $\{\partial\}$ so that $\mathbb{R}^d_+ \cup \{\partial\}$ is a one-point compactification of $\mathbb{R}^d_+$. The sub-Markov transition density of $\tilde{B}_t$ on $\mathbb{R}^d_+$ is given by $g_t(x-y) - g_t(\hat{x}-y)$. Now let us put $\tilde{X}_t = \tilde{B}_{S_t}$. The sub-Markov transition density of $\tilde{X}_t$ on $\mathbb{R}^d_+$ is given by $\int_0^\infty (g_s(x-y) - g_s(\hat{x}-y)) P(S_t \in ds) = p_t(x-y) - p_t(\hat{x}-y)$.

Next, let us consider the general case i.e. let X be a pure-jump isotropic unimodal Lévy process in $\mathbb{R}^d$ with an infinite Lévy measure $\nu(dx) = \nu(x)\, dx$ and a transition density $p_t(x)$.

For any $t > 0$, $x, y \in \mathbb{R}^d_+$ put

$$\tilde{p}_t(x,y) = p_t(x-y) - p_t(\hat{x}-y).$$

Lemma 4.1. *For any $s, t > 0$, $x, z \in \mathbb{R}^d_+$ we have*

$$\int_{\mathbb{R}^d_+} \tilde{p}_t(x,y) \tilde{p}_s(y,z)\, dy = \tilde{p}_{t+s}(x,z). \tag{17}$$

Proof. The left-hand side of (17) equals

$$\int_{\mathbb{R}^d_+} p_t(x-y) p_s(y-z)\, dy - \int_{\mathbb{R}^d_+} p_t(x-y) p_s(\hat{y}-z)\, dy - \int_{\mathbb{R}^d_+} p_t(\hat{x}-y) p_s(y-z)\, dy$$
$$+ \int_{\mathbb{R}^d_+} p_t(\hat{x}-y) p_s(\hat{y}-z)\, dy = \mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV}.$$

It is easy to check that

$$\begin{aligned} \mathrm{I} &= p_{t+s}(x-z) - \int_{\mathbb{R}^d_-} p_t(x-y) p_s(y-z)\, dy, \\ \mathrm{II} &= p_{t+s}(\hat{x}-z) - \int_{\mathbb{R}^d_-} p_t(x-y) p_s(y-\hat{z})\, dy, \\ \mathrm{III} &= \int_{\mathbb{R}^d_-} p_t(x-y) p_s(y-\hat{z})\, dy, \\ \mathrm{IV} &= \int_{\mathbb{R}^d_-} p_t(x-y) p_s(y-z)\, dy, \end{aligned}$$

which implies the lemma. □

Let C_0 be the space of all continuous functions on $\mathbb{R}^d_+$ vanishing at $\partial\mathbb{R}^d_+$ and ∞ that is $f \in C_0$ iff $f \in \mathbb{R}^d_+ \to \mathbb{R}$ is continuous and for any $\varepsilon > 0$ there exist $\delta > 0$ and $M > 0$ such that for any $x \in \mathbb{R}^d_+$ if $\mathrm{dist}(x, \partial\mathbb{R}^d_+) < \delta$ or $|x| > M$ then $|f(x)| < \varepsilon$.

For any $f \in C_0$ and $t > 0$ put $\tilde{P}_t f(x) = \int_{\mathbb{R}^d_+} \tilde{p}_t(x,y) f(y)\, dy$, $\tilde{P}_0 f(x) = f(x)$. By Lemma 4.1, $\tilde{P}_t$ is a semigroup. Extend f by putting $f(x) = -f(\hat{x})$ for $x \in \mathbb{R}^d_- = -\mathbb{R}^d_+$ and $f(x) = 0$ for $x \in \partial \mathbb{R}^d_+$. Note that $\tilde{P}_t f(x) = P_t f(x)$, $x \in \mathbb{R}^d_+$ where $P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y) f(y)\, dy$. Using this observation one can show that $\tilde{P}_t C_0 \subset C_0$ and the semigroup $\tilde{P}_t$ is strongly continuous in $t \ge 0$.

Now let us define $\tilde{P}_t(x,A)$, $t \ge 0$, $x \in \mathbb{R}^d_+$, $A \in \mathcal{B}(\mathbb{R}^d_+)$ by $\tilde{P}_t(x,A) = \int_A \tilde{p}_t(x,y)\, dy$, $t > 0$, and $\tilde{P}_0(x,\cdot) = \delta_x$. By Lemma 4.1, $\tilde{P}_t(x,A)$ is a sub-Markovian transition function on $\mathbb{R}^d_+$.

Let us augment $\mathbb{R}^d_+$ by an extra point ∂ so that $\mathbb{R}^d_+ \cup \{\partial\}$ is a one-point compactification of $\mathbb{R}^d_+$. We extend $\tilde{P}_t(x,A)$ to a Markov transition function on $\mathbb{R}^d_+ \cup \{\partial\}$ by setting

$$\tilde{P}_t(x,A) = \begin{cases} \tilde{P}_t(x, A \cap \mathbb{R}^d_+) + 1_A(\partial)(1 - \tilde{P}_t(x, \mathbb{R}^d_+)), & \text{for} \quad x \in \mathbb{R}^d_+, \\ 1_A(\partial), & \text{for} \quad x = \partial, \end{cases} \tag{18}$$

for any $A \subset \mathbb{R}^d_+ \cup \{\partial\}$ which is in the σ-algebra in $\mathbb{R}^d_+ \cup \{\partial\}$ generated by $\mathcal{B}(\mathbb{R}^d_+)$. Then by standard results (see e.g. [1, Chapter 1, Theorem 9.4]) there exists a Hunt process $\tilde{X} = (\tilde{X}_t, t \ge 0)$ with the state space $\mathbb{R}^d_+ \cup \{\partial\}$ and the transition function $\tilde{P}_t(x,A)$. We will denote by $\tilde{P}^x$, $\tilde{E}^x$ the probability and the expected value of the process $\tilde{X}$ starting from x.

Let us note $\tilde{p}_t(x,y) = \tilde{p}_t(y,x)$, $t > 0$, $x, y \in \mathbb{R}^d_+$. Put $\tau_D = \inf\{t > 0 : \tilde{X}_t \notin D\}$.

Lemma 4.2. *Let $D \subset \mathbb{R}^d_+$ be an open, nonempty set and $z \in \partial D \cap \mathbb{R}^d_+$. If there exists a cone A with vertex z such that $A \cap B(z,r) \subset D^c$ for some $r > 0$ then $\tilde{P}^z(\tau_D = 0) = 1$.*

Proof. Since $z \in \partial D \cap \mathbb{R}^d_+$ we may assume that $\text{dist}(A \cap B(z,r), \mathbb{R}^d_-) > 0$. We have under $\tilde{P}^z$, $\{\tau_D = 0\} \supset \limsup_{n\to\infty} \{\tilde{X}_{1/n} \in A \cap B(z,r)\}$. Hence

$$\tilde{P}^z(\tau_D = 0) \ge \tilde{P}^z(\limsup_{n\to\infty} \{\tilde{X}_{1/n} \in A \cap B(z,r)\}) \ge \limsup_{n\to\infty} \tilde{P}^z(\tilde{X}_{1/n} \in A \cap B(z,r)).$$

We have

$$\tilde{P}^z(\tilde{X}_{1/n} \in A \cap B(z,r)) = P^z(X_{1/n} \in A \cap B(z,r)) - P^{\hat{z}}(X_{1/n} \in A \cap B(z,r)).$$

By the rotational invariance and right-continuity of paths of X there exists $\delta = \delta(A) > 0$ such that $\limsup_{n\to\infty} P^z(X_{1/n} \in A \cap B(z,r)) \ge \delta$. Again by right-continuity of paths of X and the fact that $\text{dist}(A \cap B(z,r), \mathbb{R}^d_-) > 0$ we have $\limsup_{n\to\infty} P^{\hat{z}}(X_{1/n} \in A \cap B(z,r)) = 0$. Hence

$$\tilde{P}^z(\tau_D = 0) \ge \limsup_{n\to\infty} \tilde{P}^z(\tilde{X}_{1/n} \in A \cap B(z,r)) \ge \delta.$$

Note that the Blumenthal's zero-or-one law holds for $\tilde{X}$. Hence $\tilde{P}^z(\tau_D = 0) = 1$. $\square$

We say that $D \subset \mathbb{R}^d$ satisfies the *outer cone condition* if for any $z \in \partial D$ there exist $r > 0$ and a cone A with vertex z such that $A \cap B(z,r) \subset D^c$.

Let $D \subset \mathbb{R}^d_+$ be an open, nonempty set satisfying the outer cone condition. For any $t > 0$, $x, y \in D$ we put

$$\tilde{p}_D(t,x,y) = \tilde{p}_t(x,y) - \tilde{E}^x\left(\tilde{p}_{t-\tau_D}(\tilde{X}(\tau_D), y), t > \tau_D\right).$$

It is easy to note that for any fixed $t > 0$, $x \in D$ the function $y \to \tilde{p}_D(t,x,y)$ is continuous in $D \setminus \{x\}$. Using standard arguments (see e.g. [11, Chapter II]) one can show that for any Borel $A \subset D$, $x \in D$ and $t > 0$

$$\tilde{P}^x(\tilde{X}_t \in A, \tau_D > t) = \int_A \tilde{p}_D(t,x,y)\,dy. \tag{19}$$

Again using standard arguments and Lemma 4.2 we obtain

$$\tilde{P}^x(\tilde{X}_t \in A, \tau_D > t) = \lim_{n\to\infty} \tilde{P}^x\left(\tilde{X}_{\frac{t}{n}} \in D, \ldots, \tilde{X}_{\frac{(n-1)t}{n}} \in D, \tilde{X}_t \in A\right). \tag{20}$$

We say that a set $D \subset \mathbb{R}^d$ is symmetric if for any $x \in D$ we have $\hat{x} \in D$.

Lemma 4.3. *Assume that $D \subset \mathbb{R}^d$ is an open, symmetric, nonempty set satisfying the outer cone condition, $x \in D_+$, $0 < t_1 < \ldots < t_n$, $n \in \mathbb{N}$, $A \subset D_+$. Then we have*

$$\begin{aligned} & \tilde{P}^x\left(\tilde{X}_{t_1} \in D_+, \ldots, \tilde{X}_{t_{n-1}} \in D_+, \tilde{X}_{t_n} \in A\right) \\ = \;& P^x\left(X_{t_1} \in D, \ldots X_{t_{n-1}} \in D, X_{t_n} \in A\right) - P^{\hat{x}}\left(X_{t_1} \in D, \ldots, X_{t_{n-1}} \in D, X_{t_n} \in A\right). \end{aligned}$$

Proof. We will prove it by induction. For $n = 1$ we have

$$\tilde{P}^x\left(\tilde{X}_{t_1} \in A\right) = \int_A p_{t_1}(x-y) - p_{t_1}(\hat{x}-y)\,dy = P^x(X_{t_1} \in A) - P^{\hat{x}}(X_{t_1} \in A).$$

Assume that the assertion of the lemma holds for n, we will show it for $n+1$. Let $0 < t_1 < \ldots < t_n < t_{n+1}$. By the Markov property for $\tilde{X}_t$ we have

$$\begin{aligned} & \tilde{P}^x\left(\tilde{X}_{t_1} \in D_+, \ldots, \tilde{X}_{t_n} \in D_+, \tilde{X}_{t_{n+1}} \in A\right) \\ = \;& \tilde{E}^x\left(\tilde{X}_{t_1} \in D_+,\ \tilde{P}^{\tilde{X}_{t_1}}\left(\tilde{X}_{t_2-t_1} \in D_+, \ldots, \tilde{X}_{t_n-t_1} \in D_+, \tilde{X}_{t_{n+1}-t_1} \in A\right)\right). \end{aligned} \tag{21}$$

For any $x \in \mathbb{R}^d$ put $f(x) = P^x(X_{t_2-t_1} \in D, \ldots, X_{t_n-t_1} \in D, X_{t_{n+1}-t_1} \in A)$. By our induction hypothesis (21) equals

$$\begin{aligned} & \tilde{E}^x\left(\tilde{X}_{t_1} \in D_+,\ f\left(\tilde{X}_{t_1}\right)\right) - \tilde{E}^x\left(\tilde{X}_{t_1} \in D_+,\ f\left(\widehat{\tilde{X}_{t_1}}\right)\right) \\ = \;& \int_{D_+} (p_{t_1}(x-y) - p_{t_1}(\hat{x}-y))(f(y) - f(\hat{y}))\,dy \\ = \;& \int_{D_+} p_{t_1}(x-y) f(y)\,dy - \int_{D_+} p_{t_1}(x-y) f(\hat{y})\,dy \\ & - \int_{D_+} p_{t_1}(\hat{x}-y) f(y)\,dy + \int_{D_+} p_{t_1}(\hat{x}-y) f(\hat{y})\,dy. \end{aligned} \tag{22}$$

It is easy to verify that

$$\int_{D_+} p_{t_1}(x-y) f(\hat{y}) = \int_{D_-} p_{t_1}(\hat{x}-y) f(y)\,dy,$$

$$\int_{D_+} p_{t_1}(\hat{x}-y) f(\hat{y})\,dy = \int_{D_-} p_{t_1}(x-y) f(y)\,dy.$$

So (22) equals

$$\begin{aligned}
& \int_D p_{t_1}(x-y)f(y)\,dy - \int_D p_{t_1}(\hat{x}-y)f(y)\,dy \\
= \;& E^x\left(X_{t_1} \in D,\; f(X_{t_1})\right) - E^{\hat{x}}\left(X_{t_1} \in D,\; f(X_{t_1})\right) \\
= \;& P^x\left(X_{t_1} \in D, \ldots, X_{t_n} \in D, X_{t_{n+1}} \in A\right) - P^{\hat{x}}\left(X_{t_1} \in D, \ldots, X_{t_n} \in D, X_{t_{n+1}} \in A\right).
\end{aligned}$$

□

Let $D \subset \mathbb{R}^d$ be an open, nonempty, symmetric set satisfying the outer cone condition . Using the above lemma, (20), (19) and continuity of $y \to \tilde{p}_{D_+}(t,x,y)$ on $D \setminus \{x\}$ we obtain that for any $t > 0$, $x, y \in D_+$, we have

$$\tilde{p}_{D_+}(t,x,y) = p_D(t,x,y) - p_D(t,\hat{x},y).$$

It follows that $\tilde{p}_{D_+}(t,x,y) \le p_D(t,x,y)$.

Now let $D \subset \mathbb{R}^d$ be an open, bounded, nonempty, symmetric set. For any $x \in D_+$ we have $\tilde{E}^x(\tau_{D_+}) < \infty$. Indeed,

$$\tilde{E}^x(\tau_{D_+}) = \int_0^\infty \int_{D_+} \tilde{p}_{D_+}(t,x,y)\,dy\,dt \le \int_0^\infty \int_D p_D(t,x,y)\,dy\,dt = E^x(\tau_D) < \infty.$$

For $x, y \in D_+$ we define the Green function for $\tilde{X}_t$ and D_+ by $\tilde{G}_{D_+}(x,y) = \int_0^\infty \tilde{p}_{D_+}(t,x,y)\,dt$. For any $x, y \in D_+$, $x \ne y$ we have

$$0 < \tilde{G}_{D_+}(x,y) = G_D(x,y) - G_D(\hat{x},y) < G_D(x,y).$$

Moreover, by $\tilde{p}_{D_+}(t,x,y) \le \tilde{p}(t,x,y)$, we have a trivial bound

$$0 < \tilde{G}_{D_+}(x,y) \le \int_0^\infty \tilde{p}(t,x,y)\,dt.$$

Using standard arguments for any Borel, bounded $f : D_+ \to \mathbb{R}$ we have

$$\tilde{E}^x \int_0^{\tau_{D_+}} f(\tilde{X}_t)\,dt = \int_{D_+} \tilde{G}_{D_+}(x,y)f(y)\,dy, \quad x \in D_+.$$

For any $x, y \in D_+$, $x \ne y$ and a Borel set $A \subset \mathbb{R}^d_+$ put

$$\tilde{\nu}(x,y) = \lim_{t \to 0} \frac{\tilde{p}_t(x,y)}{t} = \nu(x-y) - \nu(\hat{x}-y)$$

and $\tilde{\nu}(x,A) = \int_A \tilde{\nu}(x,y)\,dy$. We call $\tilde{\nu}(x,A)$ the Lévy measure for the process $\tilde{X}$.

Let $D \subset \mathbb{R}^d$ be an open, bounded, nonempty, symmetric set, $x \in D_+$ and $A \subset \mathbb{R}^d_+ \setminus \overline{D}$ be a Borel set. Then by [15, Theorem 1] we have

$$\tilde{P}^x\left(\tilde{X}(\tau_{D_+}) \in A\right) = \int_{D_+} \tilde{G}_{D_+}(x,y) \int_A \tilde{\nu}(y,z)\,dz\,dy. \tag{23}$$

If additionally $\operatorname{dist}(D_+, \partial\mathbb{R}^d_+) > 0$ then again by (18) and [15, Theorem 1] we have

$$\tilde{P}^x\left(\tilde{X}(\tau_{D_+}) \in \partial\right) = \int_{D_+} \tilde{G}_{D_+}(x,y) \left(\int_{\mathbb{R}^d_-} \nu(y-z)\,dz + \int_{\mathbb{R}^d_+} \nu(\hat{y}-z)\,dz\right) dy. \tag{24}$$

Now our aim is to show that for sufficiently regular D we have $\tilde{P}^x\left(\tilde{X}(\tau_{D_+}) \in \mathbb{R}^d_+ \cap \partial D_+\right) = 0$ for $x \in D_+$.

We need to define an auxiliary family of stopping times:

$$\begin{aligned} T_{-1} &= 0, \\ T_{2n} &= \tau_{D_+} \circ \theta_{T_{2n-1}} + T_{2n-1}, \qquad n \geq 0, \\ T_{2n+1} &= \tau_{D_-} \circ \theta_{T_{2n}} + T_{2n}, \qquad n \geq 0. \end{aligned}$$

Heuristically, up to time τ_D we count consecutive jumps from D_+ to D_- and from D_- to D_+. T_0 equals the first exit time of the process from D_+, if at T_0 the process jumps to D_- then T_1 is the first exit time after T_0 from D_-. If at T_{2n} the process jumps to D_- then T_{2n+1} is the first exit time after T_{2n} from D_-. If at T_{2n+1} the process jumps to D_+ then T_{2n+2} is the first exit time of the process from D_+. If at some T_k the process jumps to D^c then all $T_m = \tau_D$ for $m \geq k$.

Lemma 4.4. *Let $D \subset \mathbb{R}^d$ be an open, bounded, nonempty, symmetric set, $x \in D_+$ and $A_+ \subset D_+$, $A_- \subset D_-$ be Borel sets. Assume that $P^y(X(\tau_D) \in \partial D) = 0$ for any $y \in D$. Then for any $n \geq 0$ we have*

$$P^x(X(T_{2n}) \in A_-) = \int_{D_+} p_{2n}(x,y) \int_{A_-} \nu(y-z)\, dz\, dy, \tag{25}$$

$$P^x(X(T_{2n+1}) \in A_+) = \int_{D_-} p_{2n+1}(x,y) \int_{A_+} \nu(y-z)\, dz\, dy, \tag{26}$$

$$E^x\left(\int_{T_{2n-1}}^{T_{2n}} 1_{A_+}(X_t)\, dt\right) = \int_{A_+} p_{2n}(x,w)\, dw, \tag{27}$$

$$E^x\left(\int_{T_{2n}}^{T_{2n+1}} 1_{A_-}(X_t)\, dt\right) = \int_{A_-} p_{2n+1}(x,w)\, dw, \tag{28}$$

where

$$p_0(x,w) = G_{D_+}(x,w),$$

$$p_{2n+1}(x,w) = \int_{D_+} p_{2n}(x,y) \int_{D_-} \nu(y-z) G_{D_-}(z,w)\, dz\, dy, \quad w \in D_-,\ n \geq 0, \tag{29}$$

$$p_{2n}(x,w) = \int_{D_-} p_{2n-1}(x,y) \int_{D_+} \nu(y-z) G_{D_+}(z,w)\, dz\, dy, \quad w \in D_+,\ n \geq 1. \tag{30}$$

Proof. We prove the lemma by induction. The case $n = 0$ is left to the Reader. Assume that (25), (26), (27), (28) hold for some $n \geq 0$. We will show it for $n+1$. By the strong Markov property we obtain

$$P^x(X(T_{2n+2}) \in A_-) = E^x\left(P^{X(T_{2n+1})}\left(X(\tau_{D_+}) \in A_-\right),\ X(T_{2n+1}) \in D_+\right). \tag{31}$$

Now the Ikeda-Watanabe formula (10) and the induction hypothesis (26) give that (31) equals

$$\begin{aligned} & E^x\left(\int_{D_+} G_{D_+}(X(T_{2n+1}),w) \int_{A_-} \nu(w-v)\, dv\, dw,\ X(T_{2n+1}) \in D_+\right) \\ = & \int_{D_-} p_{2n+1}(x,y) \int_{D_+} \nu(y-z) \int_{D_+} G_{D_+}(z,w) \int_{A_-} \nu(w-v)\, dv\, dw\, dz\, dy \\ = & \int_{D_+} p_{2n+2}(x,w) \int_{A_-} \nu(w-v)\, dv\, dw, \end{aligned}$$

which gives (25) for $n+1$. Again by the strong Markov property we get

$$
\begin{aligned}
E^x\left(\int_{T_{2n+1}}^{T_{2n+2}} 1_{A_+}(X_t)\,dt\right) &= E^x\left(\left(\left(\int_0^{\tau_{D_+}} 1_{A_+}(X_t)\,dt\right)\circ\theta_{T_{2n+1}},\, X(T_{2n+1})\in D_+\right)\right.\\
&= E^x\left(E^{X(T_{2n+1})}\left(\int_0^{\tau_{D_+}} 1_{A_+}(X_t)\,dt\right),\, X(T_{2n+1})\in D_+\right)\\
&= E^x\left(\int_{A_+} G_{D_+}(X(T_{2n+1}),w)\,dw,\, X(T_{2n+1})\in D_+\right).
\end{aligned}
$$

By the induction hypothesis (26) this is equal to

$$
\int_{D_-} p_{2n+1}(x,y)\int_{D_+}\nu(y-z)\int_{A_+} G_{D_+}(z,w)\,dw\,dz\,dy = \int_{A_+} p_{2n+2}(x,w)\,dw.
$$

This shows (27) for $n+1$. The proof of (26) and (28) for $n+1$ is analogous and it is omitted. $\square$

Lemma 4.5. *Let $D\subset\mathbb{R}^d$ be an open, bounded, nonempty, symmetric set such that $P^y(X(\tau_D)\in\partial D)=0$ for any $y\in D$. Then we have*

$$
\tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in\partial D_+\cap\mathbb{R}^d_+\right)=0,\qquad x\in D_+.
$$

Proof. First, we prove the lemma under the assumption $\operatorname{dist}(D_+,\partial\mathbb{R}^d_+)>0$. Note that $P^y(X(\tau_D)\in\partial D)=0$ for any $y\in D$ yields $P^y(X(\tau_{D_+})\in\partial D_+)=0$ for any $y\in D_+$. Moreover, our assumptions imply that the Lebesgue measure of both ∂D and ∂D_+ is zero. By (23) we have

$$
\mathrm{I}=\tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in(D^c)_+\setminus\partial D_+\right)=\int_{D_+}\tilde{G}_{D_+}(x,y)\int_{(D^c)_+}\tilde{\nu}(y,z)\,dz\,dy.
$$

By (24) we have

$$
\mathrm{II}=\tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in\{\partial\}\right)=\int_{D_+}\tilde{G}_{D_+}(x,y)\left(\int_{\mathbb{R}^d_-}\nu(y-z)\,dz+\int_{\mathbb{R}^d_+}\nu(\hat{y}-z)\,dz\right)\,dy.
$$

Hence

$$
\begin{aligned}
\mathrm{I}+\mathrm{II} &= \tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in\{\partial\}\cup(D^c)_+\setminus\partial D_+\right)\\
&= \int_{D_+}\tilde{G}_{D_+}(x,y)\int_{D^c}\nu(y-z)\,dz\,dy\\
&+ \int_{D_+}\tilde{G}_{D_+}(x,y)\left(\int_{D_-}\nu(y-z)\,dz+\int_{D_+}\nu(\hat{y}-z)\,dz\right)\,dy\\
&= \mathrm{III}+\mathrm{IV}.
\end{aligned}
$$

Note that $\mathrm{I}+\mathrm{II}=1-\tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in\partial D_+\right)$. So it is enough to show that

$$
\mathrm{III}+\mathrm{IV}=1. \tag{32}
$$

Let $X(\tau_D)_* = \lim_{t \nearrow \tau_D} X(t)$. We have

$$\begin{aligned}
\mathrm{III} &= \int_{D_+} G_D(x,y) \int_{D^c} \nu(y-z)\,dz\,dy - \int_{D_+} G_D(\hat{x},y) \int_{D^c} \nu(y-z)\,dz\,dy \\
&= \int_{D} G_D(x,y) \int_{D^c} \nu(y-z)\,dz\,dy - \int_{D_-} G_D(x,y) \int_{D^c} \nu(y-z)\,dz\,dy \\
&\qquad - \int_{D_-} G_D(\hat{x},\hat{y}) \int_{D^c} \nu(\hat{y}-z)\,dz\,dy \qquad (33) \\
&= 1 - 2\int_{D_-} G_D(x,y) \int_{D^c} \nu(y-z)\,dz\,dy \\
&= 1 - 2P^x(X(\tau_D)_* \in D_-). \qquad (34)
\end{aligned}$$

We also have

$$\begin{aligned}
\mathrm{IV} &= 2\int_{D_+} (G_D(x,y) - G_D(\hat{x},y)) \int_{D_-} \nu(y-z)\,dz\,dy \\
&= 2\left(\int_{D_+} G_D(x,y) \int_{D_-} \nu(y-z)\,dz\,dy - \int_{D_-} G_D(x,y) \int_{D_+} \nu(y-z)\,dz\,dy\right).
\end{aligned}$$

Note that by Lemma 4.4 we get

$$\begin{aligned}
G_D(x,y) &= \sum_{n=0}^{\infty} p_{2n}(x,y), \qquad \text{for } y \in D_+, \\
G_D(x,y) &= \sum_{n=0}^{\infty} p_{2n+1}(x,y), \qquad \text{for } y \in D_-.
\end{aligned}$$

It follows that

$$\mathrm{IV} = 2\left(\sum_{n=0}^{\infty} \int_{D_+} p_{2n}(x,y) \int_{D_-} \nu(y-z)\,dz\,dy - \sum_{n=0}^{\infty} \int_{D_-} p_{2n+1}(x,y) \int_{D_+} \nu(y-z)\,dz\,dy\right). \qquad (35)$$

Note that for any $z \in D_-$ we have

$$P^z(X(\tau_{D_-}) \in (D_-)^c) = \int_{D_-} G_{D_-}(z,w) \int_{(D_-)^c} \nu(w-q)\,dq\,dw = 1.$$

Using this and (29) we get

$$\begin{aligned}
&\int_{D_+} p_{2n}(x,y) \int_{D_-} \nu(y-z)\,dz\,dy \\
&= \int_{D_+} p_{2n}(x,y) \int_{D_-} \nu(y-z) \int_{D_-} G_{D_-}(z,w) \int_{(D_-)^c} \nu(w-q)\,dq\,dw\,dz\,dy \\
&= \int_{D_-} \left[\int_{D_+} p_{2n}(x,y) \int_{D_-} G_{D_-}(z,w)\nu(y-z)\,dz\,dy\right] \int_{(D_-)^c} \nu(w-q)\,dq\,dw \\
&= \int_{D_-} p_{2n+1}(x,w) \int_{(D_-)^c} \nu(w-q)\,dq\,dw \\
&= \int_{D_-} p_{2n+1}(x,w) \int_{D^c} \nu(w-q)\,dq\,dw + \int_{D_-} p_{2n+1}(x,w) \int_{D_+} \nu(w-q)\,dq\,dw.
\end{aligned}$$

Substituting this to (35) we get

$$\begin{aligned} \mathrm{IV} &= 2\left(\sum_{n=0}^{\infty}\int_{D_-} p_{2n+1}(x,w)\int_{D^c}\nu(w-q)\,dq\,dw\right) \\ &= 2\int_{D_-} G_D(x,w)\int_{D^c}\nu(w-q)\,dq\,dw \\ &= 2P^x(X(\tau_D)_*\in D_-). \end{aligned}$$

Combining the last equality with (34) we obtain (32), which completes the proof in the case $\operatorname{dist}(D_+,\partial\mathbb{R}^d_+)>0$.

To remove the above condition, for any $\varepsilon>0$, we consider $D_\varepsilon=\{(y_1,\dots,y_d)\in D_+ : y_1>\varepsilon\}$ and $H_\varepsilon=\{(y_1,\dots,y_d) : y_1>\varepsilon\}$. From (11) we infer that $P^y(X(\tau_{H_\varepsilon})\in\partial H_\varepsilon)=0$ for any $\varepsilon>0$ and $y\in H_\varepsilon$. This implies that for any $\varepsilon>0$ and $y\in D_\varepsilon$ we have $P^y(X(\tau_{D_\varepsilon})\in\partial D_\varepsilon)=0$.

Fix $x\in D_+$. There exists $\varepsilon_1>0$ such that $x\in D_{\varepsilon_1}$. For any $\varepsilon\in(0,\varepsilon_1]$ we have

$$\tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in\partial D_+\cap\mathbb{R}^d_+\right)\le\tilde{P}^x\left(\tilde{X}(\tau_{D_\varepsilon})\in\partial D_\varepsilon\cup\left(D_+\setminus\overline{D_\varepsilon}\right)\right).$$

By the first part of the proof the last probability is equal to $\tilde{P}^x\left(\tilde{X}(\tau_{D_\varepsilon})\in D_+\setminus\overline{D_\varepsilon}\right)$. By (23) we have

$$\begin{aligned} \tilde{P}^x\left(\tilde{X}(\tau_{D_\varepsilon})\in D_+\setminus\overline{D_\varepsilon}\right) &= \int_{D_\varepsilon}\tilde{G}_{D_\varepsilon}(x,y)\int_{D_+\setminus D_\varepsilon}\tilde{\nu}(y,z)\,dz\,dy \\ &\le \int_{D_\varepsilon}G_{D_\varepsilon}(x,y)\int_{D_+\setminus D_\varepsilon}\nu(y-z)\,dz\,dy \\ &\le P^x(X(\tau_{H_\varepsilon})\in\mathbb{R}^d_+\setminus H_\varepsilon). \end{aligned}$$

Clearly this tends to 0 as $\varepsilon\searrow 0$. Hence $\tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in\partial D_+\cap\mathbb{R}^d_+\right)=0$. $\square$

As a consequence of (11) and Lemma 4.5 we obtain

Corollary 4.6. *Let $D\subset\mathbb{R}^d$ be a symmetric, open, nonempty, bounded Lipschitz set. Then we have*

$$\tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in\partial D_+\cap\mathbb{R}^d_+\right)=0,\qquad x\in D_+.$$

It follows that under the assumptions of the above corollary for a Borel set $A\subset\mathbb{R}^d_+\setminus D$ and $x\in D_+$ we have

$$\tilde{P}^x\left(\tilde{X}(\tau_{D_+})\in A\right)=\int_{D_+}\tilde{G}_{D_+}(x,y)\int_A\tilde{\nu}(y,z)\,dz\,dy\le P^x\left(X(\tau_D)\in A\right). \tag{36}$$

5. Auxiliary estimates of the Lévy measure and the Green function

Throughout this section we will assume that the process X satisfies the assumptions (A). In fact it is enough to assume only (H0) and (H1).

Lemma 5.1. *For any $r>0$ we have*

$$\left|\frac{\nu'(r)}{\nu(r)}\right|\le(3(a_1-1))\,\frac{1}{r\wedge 1}.$$

Moreover, for $0<r_1<r_2<\infty$,

$$\frac{\nu(r_1)}{\nu(r_2)} \le \left(\frac{r_2}{r_1}\right)^{3(a_1-1)} e^{3(a_1-1)(r_2-r_1)}$$

and

$$\nu(r_1) - \nu(r_2) \le \frac{3}{2}(a_1 - 1)\left(\frac{\nu(r_1)}{1 \wedge r_1}\right)(r_2 - r_1)\left(1 + \frac{r_2}{r_1}\right).$$

Proof. Let $0 < u < v < \infty$. Then by absolute continuity of $\nu(\rho)$ and monotonicity of $-\nu'(\rho)/\rho$ we have

$$\nu(u) - \nu(v) = \int_u^v \rho \frac{-\nu'(\rho)}{\rho} d\rho \ge \frac{-\nu'(v)}{v}\int_u^v \rho d\rho = \frac{-\nu'(v)}{2v}(v^2 - u^2).$$

Hence

$$|\nu'(v)| \le 2v\frac{\nu(u) - \nu(v)}{v^2 - u^2}.$$

Next, we take $v = r, u = r/2$ if $r \le 2$, to arrive at

$$|\nu'(r)| \le \frac{8}{3r}(\nu(r/2) - \nu(r)) \le \frac{8(a_1 - 1)}{3r}\nu(r).$$

Similarly for $v = r + 1, u = r,\ r \ge 1$,

$$\left|\frac{\nu'(r+1)}{\nu(r+1)}\right| \le \frac{4(a_1 - 1)}{3}.$$

Combining both estimates we complete the proof of the first assertion. The second one is an easy consequence of the first.

Again let $0 < u < v < \infty$, then using monotonicity of $-\nu'(\rho)/\rho$ and the first claim of the lemma we obtain

$$\begin{aligned}
\nu(u) - \nu(v) &= \int_u^v \rho\frac{-\nu'(\rho)}{\rho}d\rho \le \frac{-\nu'(u)}{u}\int_u^v \rho d\rho \\
&= \frac{-\nu'(u)}{2u}(v^2 - u^2) \le (3(a_1 - 1))\frac{\nu(u)}{u \wedge 1}\frac{(v^2 - u^2)}{2u}.
\end{aligned}$$

□

By Lemma 5.1 and (7) we obtain

Corollary 5.2. *For any $v, z \in \mathbb{R}^d_+$,*

$$\begin{aligned}
\tilde{\nu}(v, z) &\le \frac{3}{2}(a_1 - 1)|z - \hat{z}|\frac{\nu(v - z)}{1 \wedge |v - z|}\left(1 + \frac{|v - \hat{z}|}{|v - z|}\right) \\
&\le c|z - \hat{z}|\frac{|v - \hat{z}|}{|v - z|^{d+1}(1 \wedge |v - z|)L^2(|v - z|)},
\end{aligned}$$

where $c = (a_1 - 1)c_1(d)$.

Another easy consequence of Lemma 5.1 is the following

Corollary 5.3. *If a measurable function $f : \mathbb{R}^d \to [0, \infty)$ satisfies $\int_{\mathbb{R}^d} f(y)(\nu(y) \wedge 1)\, dy < \infty$ then for any $x \in \mathbb{R}^d$ we have $\int_{\mathbb{R}^d} f(y)(\nu(x - y) \wedge 1)\, dy < \infty$.*

Lemma 5.4. *Let $w \in \mathbb{R}^d$, $r \in (0,2]$, put $B = B(w,r)$. Assume that a measurable function $f : \mathbb{R}^d \to [0,\infty)$ satisfies $\int_{\mathbb{R}^d} f(y)(\nu(y) \wedge 1)\, dy < \infty$. For $y \in B$ put $g(y) = \int_{B^c} f(z)\nu(y-z)\, dz$. Then the function g is bounded on $B(w,r/2)$ and satisfies*

$$|g(y) - g(w)| \le c\frac{(g(w) \wedge g(y))|y-w|}{r}, \qquad y \in B(w,r/2).$$

We also have $g(w) \le cg(y)$ for $y \in B(w,r/2)$.

Proof. Let $y \in B(w,r/2)$. By Corollary 5.3, $g(y) < \infty$. If $z \in B^c$, by Lemma 5.1 we have,

$$\begin{aligned} |\nu(y-z) - \nu(w-z)| &\le \nu(|z-w| - |y-w|) - \nu(|z-w|) \\ &\le 12(a_1-1)\frac{\nu\left(|z-w| - |y-w|\right)}{r}|y-w| \\ &\le \frac{\nu(|z-w|)}{r}|y-w|12(a_1-1)2^{3(a_1-1)}e^{3(a_1-1)r/2}. \end{aligned}$$

Similarly,

$$|\nu(y-z) - \nu(w-z)| \le \frac{\nu(|z-y|)}{r}|y-w|12(a_1-1)3^{3(a_1-1)}e^{3(a_1-1)r}.$$

Combining both estimates we obtain

$$|\nu(y-z) - \nu(w-z)| \le \frac{c}{r}\left(\nu(|z-y|) \wedge \nu(|w-z|)\right)|y-w|,$$

where $c = 12(a_1-1)3^{3(a_1-1)}e^{3(a_1-1)r}$.

It follows that

$$\begin{aligned} |g(y) - g(w)| &\le \int_{B^c} f(z)|\nu(y-z) - \nu(w-z)|\, dz \\ &\le c\frac{|y-w|}{r}\int_{B^c} f(z)(\nu(w-z) \wedge \nu(y-z))\, dz \\ &\le c\frac{(g(w) \wedge g(y))|y-w|}{r}. \end{aligned}$$

□

In what follows by $\{e_1, \ldots, e_d\}$ we denote the standard orthonormal basis in $\mathbb{R}^d$.

Proposition 5.5. *Let $r > 0$, $x_1 \in (0,r)$, put $B = B(0,r)$, $x = x_1e_1$. Let $y \in B_+$ such that $|y| \ge 4|x|$. Then we have*

$$0 < \tilde{G}_{B_+}(x,y) \le c|x-\hat{x}||y|U^{(d+2)}\left(\frac{|y|}{2}\right) \le c|x-\hat{x}|\frac{L^2(|y|)}{|y|^{d+1}},$$

where $c = c(d)$.

Proof. By the Lagrange theorem there is a point ξ between $\hat{x}$ and x (ξ depends on t, x, y and the process X) such that

$$p_t(x-y) - p_t(\hat{x}-y) = |x-\hat{x}|\frac{\partial}{\partial \xi_1}p_t(\xi-y).$$

By Theorem 1.5 this equals

$$-2\pi|x-\hat{x}|(\xi_1-y_1)p_t^{(d+2)}(\xi-y). \tag{37}$$

Note also that $\frac{1}{2}|y| \le |\xi - y| \le \frac{3}{2}|y|$. By Theorem 1.5, $p_t^{(d+2)}$ is radial and radially nonincreasing. Hence by (37) we have

$$|p_t(x-y) - p_t(\hat{x}-y)| \le 3\pi|x-\hat{x}||y|p_t^{(d+2)}\left(\frac{|y|}{2}\right).$$

Next,

$$\begin{aligned}\tilde{G}_{B+}(x,y) &\le \int_0^\infty |p_t(x-y) - p_t(\hat{x}-y)|\,dt \\ &\le 3\pi|x-\hat{x}||y|\int_0^\infty p_t^{(d+2)}\left(\frac{|y|}{2}\right)\,dt \\ &= 3\pi|x-\hat{x}||y|U^{(d+2)}\left(\frac{|y|}{2}\right).\end{aligned}$$

By [14, Theorem 3] the last expression is bounded from above by

$$\frac{c|x-\hat{x}|L^2\left(\frac{|x-y|}{2}\right)}{|x-y|^{d+1}} \le \frac{c|x-\hat{x}|L^2(|x-y|)}{|x-y|^{d+1}},$$

where $c = c(d)$. □

Lemma 5.6. *For any $r > 0$, $h \in (0, r/16)$, $x = he_1$, $B = B(0,r)$ we have*

$$\int_{B_+} \tilde{G}_{B_+}(x,y)|y|\,dy \le c|x|\int_{B(0,r/4)} G_B(x,y)\,dy.$$

Proof. It is obvious that

$$\int_{B(0,4h)_+} \tilde{G}_{B_+}(x,y)|y|\,dy \le 4|x|\int_{B(0,r/4)} G_B(x,y)\,dy.$$

Hence it is enough to estimate the integral over $(B \setminus B(0,4h))_+$. For any $y \in (B \setminus B(0,4h))_+$ we have $|y| \ge 4|x|$. By Proposition 5.5 we get

$$\begin{aligned}\int_{(B\setminus B(0,4h))_+} \tilde{G}_{B_+}(x,y)|y|\,dy &\le c|x-\hat{x}|\int_{(B\setminus B(0,4h))_+} |y|^2 U^{(d+2)}(|y|/2)dy \\ &\le c|x-\hat{x}|\int_0^r \rho^{d+1}U^{(d+2)}(\rho)d\rho \\ &\le c|x-\hat{x}|L^2(r),\end{aligned}$$

where the last inequality follows from [14, Proposition 2]. Finally, by Lemma 2.3 we obtain the conclusion. □

Lemma 5.7. *For any $r > 0$ and any $y \in B(0,r) \setminus \overline{B(0,3r/4)}$ we have*

$$P^y(X_{\tau_R} \in B(0,r) \setminus R) \le c\frac{L(\delta(y))}{L(r)},$$

where $R = B(0,r) \setminus \overline{B(0,r/2)}$, $\delta(y) = \delta_{B(0,r)}(y)$ and $c = c(d)$.

Proof. We may assume that $y = qe_1$ for some $q \in (3r/4, r)$. Put $z = re_1$ and $D = B(z, r/2) \cap R$. Clearly, $y \in B(z, r/4)$ and

$$P^y(X(\tau_R) \in B(0,r) \setminus R) \le P^y(X(\tau_D) \in B(0,r) \setminus D) \le P^y(X(\tau_D) \in B^c(z, r/2)).$$

By Lemma 2.4 and then by Lemma 2.3 we obtain

$$P^y(X(\tau_D) \in B^c(z, r/2)) \leq c\frac{E^y(\tau_D)}{L^2(r/2)} \leq c\frac{E^y(\tau_{B(0,r)})}{L^2(r)} \leq c\frac{L(\delta(y))}{L(r)},$$

where $c = c(d)$. □

Lemma 5.8. *For any $0 < r \leq 1$, $h \in (0, r/16)$, $x = he_1$, $B = B(0, r)$ and $y \in B_+ \setminus B(0, r/4)_+$ we have*

$$\tilde{G}_{B_+}(x, y) \leq \frac{chL(\delta(y))L(r)}{r^{d+1}},$$

where $\delta(y) = \delta_B(y)$.

Proof. Let us denote $R = B \setminus \overline{B(0, r/2)}$. By Proposition 5.5 we get

$$\tilde{G}_{B_+}(x, y) \leq \frac{chL(\delta(y))L(r)}{r^{d+1}}, \quad y \in \overline{B(0, 3r/4)} \setminus \overline{B(0, r/4)}.$$

Hence may assume that $y \in B_+ \setminus B(0, 3r/4)_+$. Since $\tilde{G}_{B_+}(x, \cdot)$ is harmonic in $B_+ \setminus \{x\}$ with respect to $\tilde{X}$ we have

$$\begin{aligned}
\tilde{G}_{B_+}(x, y) &= \tilde{E}^y\left(\tilde{G}_{B_+}\left(x, \tilde{X}(\tau_{R_+})\right)\right) \\
&= \tilde{E}^y\left(\tilde{G}_{B_+}\left(x, \tilde{X}(\tau_{R_+})\right), \tilde{X}(\tau_{R_+}) \in B_+ \setminus (R_+ \cup B(0, r/4)_+)\right) \\
&\quad + \tilde{E}^y\left(\tilde{G}_{B_+}\left(x, \tilde{X}(\tau_{R_+})\right), \tilde{X}(\tau_{R_+}) \in B(0, r/4)_+\right) = \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Since $B_+ \setminus (R_+ \cup B(0, r/4)_+$ satisfies the assumptions of Corollary 4.6 we can apply (36) to obtain

$$\begin{aligned}
\mathrm{I} &\leq \sup_{z \in B_+ \setminus (R_+ \cup B(0, r/4)_+)} \tilde{G}_{B_+}(x, z)\tilde{P}^y\left(\tilde{X}(\tau_{R_+}) \in B_+ \setminus (R_+ \cup B(0, r/4)_+)\right) \\
&\leq \sup_{z \in B_+ \setminus (R_+ \cup B(0, r/4)_+)} \tilde{G}_{B_+}(x, z)P^y\left(X_{\tau_R} \in B \setminus R\right).
\end{aligned}$$

By Proposition 5.5 and Lemma 5.7 this is bounded from above by $chL(\delta(y))L(r)r^{-d-1}$. By the Ikeda-Watanabe formula for $\tilde{X}$ (23) we get

$$\mathrm{II} = \int_{R_+} \tilde{G}_{R_+}(y, v) \int_{B(0, r/4)_+} \tilde{\nu}(v, z)\tilde{G}_{B_+}(x, z)\, dz\, dv.$$

Furthermore, by Corollary 5.2 we have for $v \in R_+$, $z \in B(0, r/4)_+$,

$$\tilde{\nu}(v, z) \leq c|z - \hat{z}|\frac{1}{|v - z|^d(1 \wedge |v - z|)L^2(|v - z|)} \leq \frac{c|z|}{r^{d+1}L^2(r)}.$$

This combined with Lemma 5.6 and the estimates of $E^x(\tau_B)$ from Lemma 2.3 yields

$$\begin{aligned}
\mathrm{II} &\leq \frac{c}{r^{d+1}L^2(r)} \int_R G_R(y, v)\, dv \int_{B_+} |z|\tilde{G}_{B_+}(x, z)\, dz \\
&\leq \frac{ch}{r^{d+1}L^2(r)} E^y(\tau_B)E^x(\tau_B) \leq \frac{chL(\delta(y))L(r)}{r^{d+1}}.
\end{aligned}$$

□

6. Proof of the main theorem

Throughout this section we will assume that the process X satisfies the assumptions (A). The following proposition is the key step in proving gradient estimates of harmonic functions for Lévy processes.

Proposition 6.1. *Let $0 < r < 1/4$, $h \in (0, r/16)$, $x = he_1$. Assume that $f : \mathbb{R}^d \to [0, \infty)$ is harmonic in $B(0, 4r)$ with respect to X. Then we have*

$$f(x) - f(\hat{x}) \le c\frac{hf(0)}{r}.$$

Proof. Put $B = B(0, r)$. For $y \in B$ put $g(y) = \int_{B^c} f(z)\nu(y - z)\, dz$. By harmonicity of f and the Ikeda-Watanabe formula (10) we have $f(x) = \int_B G_B(x, y)g(y)\, dy$. Observe $g(y) < \infty$ a.e. on B. We have

$$\begin{aligned} f(x) - f(\hat{x}) &= \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y))g(y)\, dy + \int_{B_-} (G_B(x, y) - G_B(\hat{x}, y))g(y)\, dy \\ &= \int_{B_+} (G_B(x, y) - G_B(\hat{x}, y))g(y)\, dy + \int_{B_+} (G_B(x, \hat{y}) - G_B(\hat{x}, \hat{y}))g(\hat{y})\, dy \\ &= \int_{B_+} \tilde{G}_{B_+}(x, y)(g(y) - g(\hat{y}))\, dy. \end{aligned}$$

Hence $f(x) - f(\hat{x})$ is equal to

$$\begin{aligned} &\int_{B(0,r/4)_+} \tilde{G}_{B_+}(x, y)(g(y) - g(\hat{y}))\, dy \\ &+ \int_{B_+\setminus B(0,r/4)_+} \tilde{G}_{B_+}(x, y) \int_{B^c(0,2r)} f(z)(\nu(y - z) - \nu(\hat{y} - z))\, dz\, dy \\ &+ \int_{B_+\setminus B(0,r/4)_+} \tilde{G}_{B_+}(x, y) \int_{B(0,2r)\setminus B} f(z)(\nu(y - z) - \nu(\hat{y} - z))\, dz\, dy \\ = \;& \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

By Lemma 5.4 for $y \in B(0, r/4)_+$ we obtain

$$|g(y) - g(\hat{y})| \le |g(y) - g(0)| + |g(\hat{y}) - g(0)| \le c|y|r^{-1}g(0).$$

Lemma 5.6 and the above inequality yield

$$\mathrm{I} \le \frac{cg(0)}{r} \int_{B(0,r/4)_+} \tilde{G}_{B_+}(x, y)|y|\, dy \le \frac{c|x|g(0)}{r} \int_{B(0,r/4)} G_B(x, y)\, dy.$$

Moreover, using again Lemma 5.4, we have $g(0) \le cg(y)$, $y \in B(0, r/4)$, hence

$$\mathrm{I} \le \frac{c|x|}{r} \int_{B(0,r/4)} G_B(x, y)g(y)\, dy \le \frac{c|x|f(x)}{r}.$$

II will be estimated similarly like I. For $y \in B$ put $g_1(y) = \int_{B(0,2r)^c} f(z)\nu(y - z)\, dz \le g(y)$. By Lemma 5.4 applied to g_1 we obtain

$$\mathrm{II} \le \frac{cg(0)}{r} \int_{B_+\setminus B(0,r/4)_+} \tilde{G}_{B_+}(x, y)|y|\, dy.$$

Repeating the same steps as used to estimate I we obtain

$$\mathrm{II} \le \frac{c|x|f(x)}{r}.$$

Finally we estimate III. By the assumed Harnack inequality we obtain

$$\begin{aligned}\mathrm{III} &\le cf(x)\int_{B_+\setminus B(0,r/4)_+} \tilde{G}_{B_+}(x,y)\int_{B(0,2r)\setminus B} \nu(y-z)\,dz\,dy \\ &\le cf(x)\int_{B_+\setminus B(0,r/4)_+} \tilde{G}_{B_+}(x,y)\int_{B^c(y,\delta(y))} \nu(y-z)\,dz\,dy, \qquad (38)\end{aligned}$$

where $\delta(y)=\delta_B(y)$. Denote $R(\rho)=\int_{B^c(0,\rho)}\nu(x)dx$.

By Lemma 5.8 we obtain that (38) is bounded from above by

$$\begin{aligned}&\frac{chf(x)L(r)}{r^{d+1}}\int_{B_+\setminus B(0,r/4)_+} L(\delta(y))R(\delta(y))\,dy \\ =\ &\frac{chf(x)L(r)}{r^{d+1}}\int_{r/4}^{r} \rho^{d-1}L(r-\rho)R(r-\rho)\,d\rho \\ \le\ &\frac{chf(x)L(r)}{r^{2}}\int_{0}^{r} L(\rho)R(\rho)\,d\rho \\ \le\ &\frac{chf(x)}{r}\end{aligned}$$

Here in the last step we used the estimate $\int_0^r L(\rho)R(\rho)\,d\rho \le C\frac{r}{L(r)}$ from [3, Proposition 3.5] . This gives that III is bounded from above by $chf(x)/r$.

Finally we obtain $\mathrm{I}+\mathrm{II}+\mathrm{III} \le chf(x)/r$. Using again the Harnack inequality we get $f(x)\le cf(0)$.

□

Lemma 6.2. *Let $|x|<r<|y|$. Then*

$$\left|\frac{\partial}{\partial x_1}p_t(x-y)\right| \le 6\pi\left(rp_t^{(d+2)}(r-|x|)+|y|p_t^{(d+2)}(|y|/2)\right).$$

Proof. Since $r-|x|\le|x-y|\le 3r$ for $|y|\le 2r$, and $|y|/2\le|x-y|\le 2|y|$ for $|y|>2r$, by Theorem 1.5 and radial monotonicity of $p_t^{(d+2)}$, we obtain

$$\left|\frac{\partial}{\partial x_1}p_t(x-y)\right| = 2\pi|x_1-y_1|p_t^{(d+2)}(|x-y|) \le 6\pi rp_t^{(d+2)}(r-|x|)+4\pi|y|p_t^{(d+2)}(|y|/2).$$

□

We define

$$r_D(t,x,y)=E^x(p_{t-\tau_D}(X(\tau_D),y),\, t>\tau_D), \quad x,y\in D,\, t>0.$$

Recall that $p_D(t,x,y)=p_t(x-y)-r_D(t,x,y)$.

Lemma 6.3. *For any $r\in(0,1]$, $B=B(0,r)$, $t>0$, $x,y\in B$ we have*

$$\left|\frac{\partial}{\partial x_1}r_B(t,x,y)\right| \le f_t(\delta(x),y),$$

where $f_t:(0,r]\times B\to(0,\infty)$ is a Borel function and $\delta(x)=\delta_B(x)$. For each fixed $t>0$, $y\in B$ we have $f_t(a,y)\nearrow$ when $a\searrow$ and for each $a\in(0,r]$, $y\in B$ we have

$\int_0^\infty f_t(a,y)\,dt < \infty$. *For each fixed* $t > 0$, $a \in (0,r]$ *we have* $\int_B f_t(a,y)\,dy < \infty$. *For each fixed* $a \in (0,r]$ *we have* $\int_0^\infty \int_B f_t(a,y)\,dy\,dt < \infty$.

Proof. We have

$$\left|\frac{\partial}{\partial x_1} r_B(t,x,y)\right| = \left|\frac{\partial}{\partial x_1} E^y\left[p_{t-\tau_B}(x - X(\tau_B)), t > \tau_B\right]\right|. \tag{39}$$

Applying Lemma 6.2 we obtain

$$\left|\frac{\partial}{\partial x_1}\left[p_{t-\tau_B}(x - X(\tau_B))\right]\right| \le 6\pi r p^{(d+2)}_{t-\tau_B}(r - |x|) + 6\pi |X(\tau_B)| p^{(d+2)}_{t-\tau_B}(|X(\tau_B)|/2)).$$

Moreover by Corollary 1.7,

$$\left|\frac{\partial}{\partial x_1}\left[p_{t-\tau_B}(x - X(\tau_B))\right]\right| \le \frac{c}{(r - |x|)^{d+1}}.$$

It follows that we can change the order of $\frac{\partial}{\partial x_1}$ and E^y in (39). We have also shown that

$$\left|\frac{\partial}{\partial x_1} r_B(t,x,y)\right| \le f_t(\delta(x), y),$$

where

$$f_t(a,y) = 6\pi E^y\left[r p^{(d+2)}_{t-\tau_B}(a), t > \tau_B\right] + 6\pi E^y\left[|X(\tau_B)| p^{(d+2)}_{t-\tau_B}(|X(\tau_B)|/2), t > \tau_B\right].$$

Of course, $f_t : (0,r] \times B \to (0,\infty)$, f_t is a Borel function, for each fixed $t > 0$, $y \in B$ we have $f_t(a,y) \nearrow$ when $a \searrow$. We also have

$$\begin{aligned}
\int_0^\infty f_t(a,y)\,dt &= 8\pi r E^y\left[\int_{\tau_B}^\infty p^{(d+2)}_{t-\tau_B}(a)\,dt\right] \\
&\quad + 4\pi E^y\left[|X(\tau_B)| \int_{\tau_B}^\infty p^{(d+2)}_{t-\tau_B}(|X(\tau_B)|/2)\,dt\right] \\
&= 6\pi r U^{(d+2)}(a) + 6\pi E^y\left[|X(\tau_B)| U^{(d+2)}(|X(\tau_B)|/2)\right] \\
&\le 8\pi r U^{(d+2)}(a) + 8\pi \sup_{\rho > r} \rho U^{(d+2)}(\rho).
\end{aligned}$$

By [14, Theorem 16] and then by (5) this is bounded from above by

$$\frac{crL^2(a)}{a^{d+2}} + \sup_{\rho > r} \frac{cL^2(\rho)}{\rho^{d+1}} < \frac{crL^2(a)}{a^{d+2}} + \sup_{\rho > r} \frac{c\frac{\rho^2}{r^2} L^2(r)}{\rho^{d+1}} \le \frac{crL^2(a)}{a^{d+2}} + \frac{cL^2(r)}{r^{d+1}},$$

where $c = c(d)$.

It follows that for each fixed $a \in (0,r]$ we have $\int_0^\infty \int_B f_t(a,y)\,dy\,dt < \infty$. □

By saying that $\frac{\partial f}{\partial x_i}(x)$ exists we understand that $\lim_{h \to 0} \frac{f(x + he_i) - f(x)}{h}$ exists and is finite.

Lemma 6.4. *For any* $r \in (0,1]$, $B = B(0,r)$, $x, y \in B$, $x \ne y$ *there exists*

$$\frac{\partial}{\partial x_1} G_B(x,y)$$

and

$$\frac{\partial}{\partial x_1} G_B(x,y) = \int_0^\infty \frac{\partial}{\partial x_1} p_B(t,x,y)\,dt.$$

Proof. Fix $t > 0$, $x, y \in B$, $x \neq y$ and put $s = \frac{|x-y|}{2} \wedge \frac{\delta(x)}{2}$. We will estimate $\frac{\partial}{\partial z_1} p_B(t,z,y)$ for $z \in B(x,s)$. For $z \in B(x,s)$ we have

$$\frac{\partial}{\partial z_1} p_B(t,z,y) = \frac{\partial}{\partial z_1} p_t(z-y) - \frac{\partial}{\partial z_1} r_B(t,z,y).$$

We have

$$\left| \frac{\partial}{\partial z_1} p_t(z-y) \right| = 2\pi |z_1 - y_1| p_t^{(d+2)}(|z-y|) \leq 4\pi r p_t^{(d+2)}(s)$$

and

$$\left| \frac{\partial}{\partial z_1} r_B(t,z,y) \right| \leq f_t(\delta(z), y) \leq f_t(s,y),$$

where f_t is a function defined in Lemma 6.3. We have $\int_0^\infty (p_t^{(d+2)}(s) + f_t(s,y))\, dt < \infty$. This justifies the change of the derivative and integral in $\frac{\partial}{\partial x_1} \int_0^\infty p_B(t,x,y)\, dt$ and implies the assertion of the lemma. □

Proposition 6.5. *For any $r \in (0,1]$, $B = B(0,r)$, $x, y \in B$, $x \neq y$ we have*

$$\frac{\partial}{\partial x_1} G_B(x,y) \leq c \frac{G_B(x,y)}{|x-y| \wedge \delta(x)},$$

where $\delta(y) = \delta_B(y)$.

Proof. Fix $x, y \in B$, $x \neq y$. Let $s = |x-y| \wedge \delta(x)$. The function $z \to G_B(z,y)$ is harmonic (with respect to the process X) for $z \in B(x, s/2)$. By continuity of $z \to G_B(z,y)$, $z \neq y$ we obtain that there exists $\varepsilon > 0$ such that for $|x-z| < \varepsilon$ we have $G_B(z,y) \leq 2G_B(x,y)$. Let $h \in (0, \varepsilon \wedge (s/4))$. By Proposition 6.1 we have

$$\left| \frac{G_B(x+he_1,y) - G_B(x,y)}{h} \right| \leq c \frac{G_B\left(x + \frac{1}{2}he_1, y\right)}{s}. \tag{40}$$

Since $h \leq \varepsilon$ we obtain $G_B\left(x + \frac{1}{2}he_1, y\right) \leq 2G_B(x,y)$. Using this, (40) and Lemma 6.4 we obtain the assertion of the lemma. □

Lemma 6.6. *For any $R \in (0,1]$, $B = B(0,R)$, $x \in B$ the partial derivative $\frac{\partial}{\partial x_1} E^x(\tau_B)$ exists. Moreover it equals to 0 for $x = 0$.*

Proof. We begin the proof with an extra assumtion that for any $t > 0$, $p_t^{(d+2)}(0) = \|p_t^{(d+2)}\|_\infty < \infty$, where $\|\cdot\|_\infty$ denotes the supremum norm. Fix $z \in B$ and put $4r = \delta_B(z)$. We will show that for any $x \in B(z,r)$, $\frac{\partial}{\partial x_1} E^x(\tau_B)$ exists.

For any $t > 0$, $x \in B(z,r)$ put $g_t(x) = \int_{B(z,2r)} p_t(x-y)\, dy$. For any $t > 0$, $x \in B(z,r)$, $y \in B(z,2r)$ we have

$$\left| \frac{\partial}{\partial x_1} p_t(x-y) \right| = 2\pi \left| (x_1 - y_1) p_t^{(d+2)}(x-y) \right| \leq 6\pi r \|p_t^{(d+2)}\|_\infty < \infty.$$

It follows that for any $t > 0$, $x \in B(z,r)$ we have

$$\frac{\partial g_t}{\partial x_1}(x) = \int_{B(z,2r)} \frac{\partial p_t}{\partial x_1}(x-y)\, dy = -2\pi \int_{B(z,2r)} (x_1 - y_1) p_t^{(d+2)}(x-y)\, dy.$$

In particular, $\frac{\partial g_t}{\partial x_1}(x)$ exists for any $x \in B(z,2r)$.

For any $t > 0$, $x \in B(z, r)$ we also have

$$\frac{\partial g_t}{\partial x_1}(x) = -2\pi \int_{B(x,r)} (x_1 - y_1) p_t^{(d+2)}(x-y)\, dy - 2\pi \int_{B(z,2r)\setminus B(x,r)} (x_1 - y_1) p_t^{(d+2)}(x-y)\, dy.$$

By radial symmetry of $p_t^{(d+2)}$ the first integral vanishes, hence finally

$$\frac{\partial g_t}{\partial x_1}(x) = -2\pi \int_{B(z,2r)\setminus B(x,r)} (x_1 - y_1) p_t^{(d+2)}(x-y)\, dy. \tag{41}$$

To remove the assumption that $p_t^{(d+2)}(0) = \|p_t^{(d+2)}\|_\infty < \infty$ we consider the process with the symbol $\psi(\xi) + \epsilon|\xi|$, $\epsilon > 0$, that is we add to X an independent Cauchy process multiplied by $\epsilon > 0$. This new process satisfies all the assumptions needed in Theorem 1.5 to construct its $d+2$-dimensional corresponding variant. Moreover this $d+2$ - dimensional process has uniformly bounded transition densities for each t. Hence we can repeat all the above steps and then pass with $\epsilon \to 0$ to arrive at (41) in this case. The passage is easily justyfied by observing that the integrand in (41) is bounded by $3rp_t^{(d+2)}(r)$. We leave the details to the Reader.

Hence for any $t > 0$, $x \in B(z, r)$ we have

$$\begin{aligned}
\left|\frac{\partial g_t}{\partial x_1}(x)\right| &= 2\pi \left| \int_{B(z,2r)\setminus B(x,r)} (x_1 - y_1) p_t^{(d+2)}(x-y)\, dy \right| \\
&\le 6\pi \int_{B(x,3r)\setminus B(x,r)} |x-y| p_t^{(d+2)}(x-y)\, dy \\
&\le \frac{c}{r} \int_r^{3r} \rho^{d+1} p_t^{(d+2)}(\rho)\, d\rho \\
&= \frac{c}{r} \int_{B^*(0,3r)\setminus B^*(0,r)} p_t^{(d+2)}(y)\, dy,
\end{aligned}$$

where $c = c(d)$ and $B^*(0, u) = \{y \in \mathbb{R}^{d+2} : |y| < u\}$, $u > 0$.

Now for any $t > 0$, $x \in B(z, r)$ put $h_t(x) = \int_{B(z,2r)} r_B(t, x, y)\, dy$. By Lemma 6.3 for any $t > 0$, $x \in B(z, r)$, $y \in B(z, 2r)$ we have

$$\left| \frac{\partial}{\partial x_1} r_B(t, x, y) \right| \le f_t(\delta(x), y) \le f_t(3r, y).$$

For any $t > 0$ Lemma 6.3 also gives $\int_B f_t(3r, y)\, dy < \infty$. Hence for any $t > 0$, $x \in B(z, r)$ we have

$$\frac{\partial h_t}{\partial x_1}(x) = \int_{B(z,2r)} \frac{\partial}{\partial x_1} r_B(t, x, y)\, dy$$

and

$$\left|\frac{\partial h_t}{\partial x_1}(x)\right| \le \int_{B(z,2r)} f_t(3r, y)\, dy < \infty.$$

Now for any $x \in B(z, r)$ put

$$A(x) = \int_{B(z,2r)} G_B(x, y)\, dy = \int_0^\infty (g_t(x) - h_t(x))\, dt. \tag{42}$$

Now for any $t > 0$, $x \in B(z, r)$ we have

$$\left| \int_0^\infty \left(\frac{\partial g_t}{\partial x_1}(x) - \frac{\partial h_t}{\partial x_1}(x) \right) dt \right| \le \frac{c}{r} \int_{B^*(0,3r)\setminus B^*(0,r)} p_t^{(d+2)}(y)\, dy + \int_{B(z,2r)} f_t(3r, y)\, dy.$$

Next,

$$\frac{c}{r}\int_0^\infty \int_{B^*(0,3r)\setminus B^*(0,r)} p_t^{(d+2)}(y)\,dy\,dt = \frac{c}{r}\int_{B^*(0,3r)\setminus B^*(0,r)} U^{(d+2)}(y)\,dy < \infty.$$

By Lemma 6.3 we have

$$\int_0^\infty \int_{B(z,2r)} f_t(3r,y)\,dy\,dt < \infty.$$

Using this and (42) we obtain that $\frac{\partial}{\partial x_1}A(x)$ exists for any $x \in B(z,r)$.

Now for any $x \in B(z,r)$ put

$$B(x) = \int_{B\setminus B(z,2r)} G_B(x,y)\,dy.$$

By Proposition 6.5 for any $x \in B(z,r)$, $y \in B \setminus B(z,2r)$ we get

$$\left|\frac{\partial}{\partial x_1}G_B(x,y)\right| \le c\frac{G_B(x,y)}{r} \le c\frac{G_B(0,y)}{r},$$

where the last step follows by applying the Harnack inequality to $G_B(\cdot,y)$. Since $G_B(0,y)$ is integrable we obtain that $\frac{\partial}{\partial x_1}B(x)$ exists for any $x \in B(z,r)$. Finally the function $x \to E^x(\tau_B)$ is symmetric, so its partial derivative exists at $x = 0$ and must be equal to 0. The proof is completed. □

Proposition 6.7. *Let $f : \mathbb{R}^d \to [0,\infty)$ be harmonic in an open nonempty set $D \subset \mathbb{R}^d$. Then $\frac{\partial f}{\partial x_i}(x)$ exists for any $i = 1,\ldots,d$ and $x \in D$.*

Proof. The proof resembles to some extent the proof of Lemma 4.3 in [20]. We may assume that $i = 1$. Fix $z \in D$ and choose $r \in (0,1]$ such that $B(z,3r) \subset D$. Put $B = B(z,r)$. Then we have

$$f(x) = \int_B G_B(x,y)\int_{B^c} f(w)\nu(y-w)\,dw\,dy, \qquad x \in B.$$

Put

$$g(y) = \int_{B^c} f(w)\nu(y-w)\,dw, \qquad y \in B$$

and $u(y) = g(y) - g(z)$, $y \in B$. We have $f(x) = G_B g(x)$. By Lemma 5.4 we obtain

$$|u(y)| \le \frac{c}{r}|y-z|g(z), \qquad y \in B(z,r/2). \tag{43}$$

Let $h \in (-r/8, r/8)$. We have

$$\begin{aligned} G_B g(z+he_1) - G_B g(z) &= (G_B 1_B(z+he_1) - G_B 1_B(z))g(z) \\ &\quad + G_B u(z+he_1) - G_B u(z). \end{aligned}$$

By Lemma 6.6 we get

$$\lim_{h\to 0}\frac{1}{h}(G_B 1_B(z+he_1) - G_B 1_B(z))g(z) = g(z)\frac{\partial}{\partial z_1}G_B 1_B(z) = 0.$$

We also have

$$\frac{1}{h}(G_B u(z+he_1) - G_B u(z)) = \int_B \frac{1}{h}(G_B(z+he_1,y) - G_B(z,y))u(y)\,dy$$

$$= \int_{B(z,2|h|)} + \int_{B\setminus B(z,2|h|)} = \mathrm{I} + \mathrm{II}.$$

By (43) and next by the bounded convergence theorem

$$\begin{aligned} |\mathrm{I}| &\le cg(z)\int_{B(z,2|h|)} \left(G_B(z+he_1,y)+G_B(z,y)\right)\, dy. \\ &\le cg(z)\int_{B(0,3|h|)} G_{B(0,2r)}(0,y)\, dy \to 0, \quad h\to 0. \end{aligned}$$

Applying Proposition 6.5 we have for any $y \in B \setminus B(z, 2|h|)$,

$$\begin{aligned} \left|\frac{1}{h}(G_Bu(z+he_1)-G_Bu(z))\right| &= \left|\frac{\partial G_B}{\partial z_1}(z+h\theta e_1,y)\right| \\ &\le c\frac{G_B(z+h\theta e_1,y)}{|z-y|\wedge r} \\ &\le c\frac{G_B(z,y)}{|z-y|\wedge r}, \end{aligned}$$

where $0 \le \theta \le 1$ and the last inequality follows from the Harnack principle. Next we show that $\frac{G_B(z,y)}{|z-y|\wedge r}|u(y)|$ is intgrable over B. By (43), for $y \in B(z, r/2)$, we have

$$\frac{G_B(z,y)}{|z-y|\wedge r}|u(y)| \le c\frac{G_B(z,y)}{r|z-y|}g(z)|z-y| = c\frac{G_B(z,y)}{r}g(z),$$

while for $y \in B \setminus B(z, r/2)$ we obtain

$$\frac{G_B(z,y)}{|z-y|\wedge r}|u(y)| \le \frac{G_B(z,y)}{|z-y|\wedge r}(g(z)+g(y)) \le 2\frac{G_B(z,y)}{r}(g(z)+g(y)).$$

Of course, $y \to G_B(z,y)g(y) = G_B(z,y)\int_{B^c} f(w)\nu(y-w)\, dw$ is an integrable function on B. This implies

$$\lim_{h\to 0} \mathrm{II} = \int_B \frac{\partial}{\partial z_1}G_B(z,y)u(y)\, dy$$

and finishes the proof of the proposition. □

Proof of Theorem 1.1. Fix $z \in D$ and let $r = (\delta_D(z)/3) \wedge (1/4)$. We may assume that $z = 0$. Note that it is enough to consider $\frac{\partial f}{\partial x_1}$. By Proposition 6.7 $\frac{\partial f}{\partial x_1}(0)$ exists and Proposition 6.1 implies (1). □

7. Examples

The processes in the first 3 examples are subordinate Brownian motions in $\mathbb{R}^d$, i.e. $X_t = B_{S_t}$ where B is the Brownian motion in $\mathbb{R}^d$ (with a generator Δ) and S is an independent subordinator with the Laplace exponent ϕ.

Example 7.1. We assume that the Levy measure of the subordinator S is infinite, ϕ is a complete Bernstein function and it satisfies

$$c_1\lambda^{\alpha/2}\ell(\lambda) \le \phi(\lambda) \le c_2\lambda^{\alpha/2}\ell(\lambda), \qquad \lambda \ge 1,$$

where $0 < \alpha < 2$, ℓ varies slowly at infinity, i.e. $\forall x > 0\ \lim_{\lambda\to\infty} \frac{\ell(\lambda x)}{\ell(\lambda)} = 1$. The process X satisfies assumptions (A).

In particular, one of the processes satisfying the above conditions is *the relativistic process* in $\mathbb{R}^d$ with the Laplace exponent $\phi(\lambda) = \sqrt{\lambda + m^2} - m$, $m > 0$ and a generator $m - \sqrt{m^2 - \Delta}$, (see [7], [25], [21]). The generator of this process is called the relativistic Hamiltonian and it is used in some models of mathematical physics (see e.g. [22]).

Proof. It is clear that assumptions (H4), (H10) are satisfied. The fact that (H7) holds it is stated in Example 4 in [17]. Hence assumptions (A3) are satisfied. □

Example 7.2. Let $\phi(\lambda) = \log(1 + \lambda^{\beta/2})$, $\beta \in (0, 2]$. The process X is called *the geometric stable processes* and it satisfies assumptions (A).

Proof. One can directly check that (H10) is satisfied (see also Example 1 in [17]), the fact that (H7) holds is well known, (H4) is obvious. Hence assumptions (A3) are satisfied. □

Example 7.3. Let $\phi(\lambda) = \frac{\lambda}{\log(1+\lambda)} - 1$. The process X is sometimes called *the conjugate to the variance gamma process* and it satisfies assumptions (A).

Proof. (H0) is clear, (H2) follows from [24] (for $d \geq 3$) and [14, Example 3] ($d \geq 1$). (H1) is implied by two conditions which hold for the density of the Lévy measure of the subordinator ν_S (see the proof of Proposition 3.5 [18]),

(a) For any $K > 0$ there is $c = c(K)$ such that

$$\nu_S(r) \leq c\nu_S(2r), \ 0 < r < K.$$

(b) There exists C such that

$$\nu_S(r) \leq C\nu_S(r + 1), \ r \geq 1.$$

From the estimates of $\nu_S(r)$ obtained in [24] we infer that (a) holds, while (b) is implied by the fact that $\phi(\lambda)$ is a complete Bernstein function [18, Lemma 2.1]. □

The process in the next example is not a subordinate Brownian motion.

Example 7.4. Let $\{X_t\}$ be the pure-jump isotropic Lévy process in $\mathbb{R}^d$ with the Lévy measure $\nu(dx) = \nu(|x|)\, dx$ given by the formula

$$\nu(r) = \begin{cases} \mathcal{A}_{d,\alpha} r^{-d-\alpha} & \text{for} \quad r \in (0, 1] \\ c_1 e^{-c_2 r} & \text{for} \quad r \in (1, \infty) \end{cases}$$

where $\mathcal{A}_{d,\alpha} r^{-d-\alpha}$ is the Lévy density for the symmetric α-stable process in $\mathbb{R}^d$ with the characteristic exponent $\psi(x) = |x|^\alpha$, $\alpha \in (0, 2)$ and $c_1 = \mathcal{A}_{d,\alpha} e^{d+\alpha} > 0$, $c_2 = d + \alpha > 0$ are chosen so that $\nu(r) \in C^1(0, \infty)$. X satisfies assumptions (A).

Proof. (H0) is obvious and (H1) is easy to check. (WLSC) holds for $\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos\langle \xi, x\rangle)\, \nu(dx)$ because the characteristic exponent ψ for X behaves for large ξ like the characteristic exponent for the symmetric α-stable process. Hence (H3) holds, so assumptions (A1) are satisfied. □

Now we show an example of a harmonic function for some Lévy process for which the gradient does not exist at some point. The process is a pure-jump, isotropic unimodal Lévy process, which Lévy measure does not satisfy the assumption that $-\nu'(r)/r$ is nonincreasing (cf. (H1)).

Example 7.5. Let X be a pure-jump, Lévy process in $\mathbb{R}$ which Lévy measure $\nu(dx) = \nu(x)\,dx$ has the density given by the formula

$$\nu(x) = \begin{cases} \mathcal{A}_\alpha |x|^{-1-\alpha} & \text{for} \quad |x| \in (0,1], \\ \mathcal{A}_\alpha (1 - (|x|-1)^\gamma) & \text{for} \quad |x| \in (1,2], \\ 0 & \text{for} \quad |x| \in (2,\infty), \end{cases}$$

where $\alpha \in (0,1/2)$, $\gamma \in (1/2,1)$, $\alpha + \gamma < 1$, $\mathcal{A}_\alpha |x|^{-1-\alpha}$ is the density of the Lévy measure for α-stable process in $\mathbb{R}$ with the characteristic exponent $\psi(x) = |x|^\alpha$.

Note that $\nu(x)$ satisfies $\nu(-x) = \nu(x)$, it is continuous and nonincreasing on $(0,\infty)$. It follows that the process X is isotropic unimodal.

Let $B = (-1/2, 1/2)$ and let us define the function f by

$$f(z) = \begin{cases} (z-1)^{-\beta} & \text{for} \quad z \in (1,2), \\ 0 & \text{for} \quad z \notin (B \cup (1,2)), \\ E^z(f(X(\tau_B))) & \text{for} \quad z \in B, \end{cases}$$

where $\beta \in (0,1)$, $\alpha - \beta + \gamma < 0$.

Then f is harmonic on B with respect to the process X but $f'(0)$ does not exist.

Proof. Note that for any $y \in B_+$ we have $\tilde{P}^y(\tilde{X}(\tau_{B_+}) \in [1/2, 5/2]) = 1$. By the arguments used in the proof of Proposition 6.1 we get

$$f(x) - f(-x) = \int_{B_+} \tilde{G}_{B_+}(x,y)(g(y) - g(-y))\,dy, \tag{44}$$

where $B_+ = (0,1/2)$, $x \in B_+$, $g(y) = \int_{B^c} \nu(y-z) f(z)\,dz$, $y \in \mathbb{R}$. For $y \in B_+$ we have

$$\begin{aligned} g(y) &= \mathcal{A}_\alpha \int_1^{1+y} (z-y)^{-1-\alpha}(z-1)^{-\beta}\,dz + \mathcal{A}_\alpha \int_{1+y}^2 (1-(z-y-1)^\gamma)(z-1)^{-\beta}\,dz \\ &\geq \mathcal{A}_\alpha \int_1^2 (z-1)^{-\beta}\,dz - \mathcal{A}_\alpha \int_{1+y}^2 (z-y-1)^\gamma (z-1)^{-\beta}\,dz \end{aligned}$$

and

$$g(-y) = \mathcal{A}_\alpha \int_1^2 (z-1)^{-\beta}\,dz - \mathcal{A}_\alpha \int_1^2 (z+y-1)^\gamma (z-1)^{-\beta}\,dz.$$

Hence for $y \in B_+$ we have

$$g(y) - g(-y) \geq \mathcal{A}_\alpha \int_1^{1+y} (z+y-1)^\gamma (z-1)^{-\beta}\,dz \geq c y^{1-\beta+\gamma}, \tag{45}$$

where $c = c(\alpha,\beta,\gamma)$.

Now we need to use the inequality, which we justify later:

$$\tilde{G}_{B_+}(x,y) \geq c_1 x y^{\alpha-2} - c_2 x, \qquad x \in (0,1/16),\ y \in (2x, 1/4), \tag{46}$$

where $c_1 = c_1(\alpha,\gamma)$, $c_2 = c_2(\alpha,\gamma)$. Using (44), (45) and (46) we get for $x \in (0,1/16)$

$$f(x) - f(-x) \geq c_1 x \int_{2x}^{1/4} y^{\alpha-\beta+\gamma-1}\,dy - c_2 x \geq c_1 x^{1+\alpha-\beta+\gamma} - c_2 x,$$

where $c_1 = c_1(\alpha,\beta,\gamma)$, $c_2 = c_2(\alpha,\beta,\gamma)$. By our assumptions on α, β, γ we obtain $1 + \alpha - \beta + \gamma \in (0,1)$. It follows that $f'(0)$ does not exist.

What remains is to prove (46). By the definition of $\tilde{G}_{B_+}(x,y)$ we have

$$\begin{aligned}\tilde{G}_{B_+}(x,y) &\geq \int_0^1 \tilde{p}_{B_+}(t,x,y)\,dt & (47)\\ &= \int_0^1 \tilde{p}(t,x,y) - \tilde{E}^y\left(\tilde{p}(t-\tau_{B_+},x,\tilde{X}(\tau_{B_+})),\, \tau_{B_+}<t\right)\,dt. & (48)\end{aligned}$$

Let ψ be the characteristic exponent of X. By the formula for ν we obtain that ψ satisfies WLSC, so we can use Proposition 3.1. By this proposition there exists a Lévy process in $\mathbb{R}^3$ with the characteristic exponent $\psi^{(3)}(\xi)=\psi(|\xi|)$, $\xi\in\mathbb{R}^3$ and the continuous transition density $p_t^{(3)}(x)=p_t^{(3)}(|x|)$, $x\in\mathbb{R}^3$, $t>0$ satisfying $p_t^{(3)}(r)=(-1/(2\pi r))p_t'(r)$, $r>0$. It follows that

$$\tilde{p}(t,x,y)=-2xp_t'(y+\xi)=4\pi x(y+\xi)p_t^{(3)}(y+\xi), \tag{49}$$

$$\tilde{p}(t-\tau_{B_+},x,\tilde{X}(\tau_{B_+}))=-2xp'_{t-\tau_{B_+}}(\tilde{X}(\tau_{B_+})+\xi)=4\pi x(\tilde{X}(\tau_{B_+})+\xi)p^{(3)}_{t-\tau_{B_+}}(\tilde{X}(\tau_{B_+})+\xi), \tag{50}$$

where $\xi\in(-x,x)$.

Let $D\subset\mathbb{R}$ be a bounded, open, nonempty, symmetric ($D=-D$) set. For any $s>0$, $y\in D_+$, $z\in\left((\overline{D})^c\right)_+$ put

$$h_{D_+}(y,s,z)=\int_{D_+}\tilde{p}_{D_+}(s,y,w)\tilde{\nu}(w-z)\,dw. \tag{51}$$

By (23) and standard arguments (see e.g. Proposition 2.5 in [21]) $h_{D_+}(y,s,z)$ provides the distribution of $(\tau_{D_+},\tilde{X}(\tau_{D_+}))$ if the process $\tilde{X}$ starts from $y\in D_+$.

Note that

$$h_{D_+}(y,s,z)\leq \sup_{w\in D_+}\nu(w-z)\int_{D_+}\tilde{p}_{D_+}(s,y,w)\,dy\leq \sup_{w\in D_+}\nu(w-z),$$

$$\begin{aligned}&\tilde{E}^y\left(\tilde{p}(t-\tau_{B_+},x,\tilde{X}(\tau_{B_+})),\,\tau_{B_+}<t,\,\tilde{X}(\tau_{B_+})>3/4\right) & (52)\\ =\;& \int_0^t\int_{3/4}^{5/2} h_{D_+}(y,s,z)\tilde{p}(t-s,x,z)dzds & (53)\\ \leq\;& c\int_0^t\int_{3/4}^{5/2}\tilde{p}(s,x,z)dzds = c\int_0^t\int_{3/4}^{5/2}\int_{-x}^{x}\frac{d}{dw}p(s,w-z)dwdzds & (54)\\ =\;& c\int_0^t\int_{-x}^{x}\int_{3/4}^{5/2}(z-w)p^{(3)}(s,w-z)dzdwds\leq ctx, & (55)\end{aligned}$$

where $c=c(\alpha,\gamma)$.

Let Y be the symmetric α-stable process in $\mathbb{R}^3$, with the Lévy measure $\nu_Y^{(3)}(dx)=\nu_Y^{(3)}(|x|)\,dx$ and the transition density $q_t^{(3)}(x)$. Now we need to use the inequality, which we justify later:

$$|p_t^{(3)}(y)-e^{-mt}q_t^{(3)}(y)|\leq c,\qquad t\in(0,1),\, y\in B(0,13/16), \tag{56}$$

where $c=c(\alpha,\gamma)$, $m=m(\alpha,\gamma)\in(-\infty,\infty)$. It is well known that

$$c_1\min(t^{-3/\alpha},t|y|^{-3-\alpha})\leq q_t^{(3)}(y)\leq c_2\min(t^{-3/\alpha},t|y|^{-3-\alpha}),\quad t>0,\, y\in\mathbb{R}^3, \tag{57}$$

where $c_1=c_1(\alpha)$, $c_2=c_2(\alpha)$.

By (49) and (56) we get for $x \in (0, 1/16)$, $y \in (2x, 1/4)$, $t \in (0, 1]$

$$\tilde{p}(t, x, y) \ge c_1 x y q_t^{(3)}(y) - c_2 x, \tag{58}$$

where $c_1 = c_1(\alpha, \gamma)$, $c_2 = c_2(\alpha, \gamma)$.

By (50), (56) and (57) we get for $x \in (0, 1/16)$, $y \in (2x, 1/4)$, $t \in (0, 1]$

$$\tilde{E}^y \left(\tilde{p}(t - \tau_{B_+}, x, \tilde{X}(\tau_{B_+})), \, \tau_{B_+} < t, \, \tilde{X}(\tau_{B_+}) \le 3/4 \right) \le cx, \tag{59}$$

where $c = c(\alpha, \gamma)$.

By (47 - 48), (58), (52 - 55) and (59) we get for $x \in (0, 1/16)$, $y \in (2x, 1/4)$

$$\tilde{G}_{B_+}(x, y) \ge c_1 x y \int_0^1 q_t^{(3)}(y) \, dt - c_2 x \ge c_1 x y^{\alpha - 2} - c_2 x,$$

where $c_1 = c_1(\alpha, \gamma)$, $c_2 = c_2(\alpha, \gamma)$. This gives (46).

What remains is to show (56). Denote by $\nu^{(3)}(dx) = \nu^{(3)}(|x|)$ the Lévy meausure of $X^{(3)}$. By (2) for $r \in (0, \infty) \setminus \{1, 2\}$ we get

$$\nu^{(3)}(r) = 1_{(0,1)}(r) \nu_Y^{(3)}(y) - 1_{(1,2)}(r) \frac{\nu'(r)}{2\pi r} = \nu_Y^{(3)}(y) + \mu(r), \tag{60}$$

where $\operatorname{supp}(\mu) = [1, \infty)$, $\mu(r)$ has the singularity at $r = 1$ of the type $(r - 1)^{\gamma - 1}$ and $\mu(r)$ changes the sign on $(1, \infty)$. Put $\mu(x) = \mu(|x|)$, $x \in \mathbb{R}^3$, $m = \int_{\mathbb{R}^3} \mu(x) \, dx$, $M = \int_{\mathbb{R}^3} |\mu(x)| \, dx$. We have $M < \infty$, $m \in (-\infty, \infty)$. From (60) it follows that for $t > 0$

$$p_t^{(3)} = q_t^{(3)} * \left(e^{-tm} \sum_{n=0}^{\infty} \frac{t^n \mu^{(*n)}}{n!} \right)$$

so

$$p_t^{(3)}(x) = e^{-tm} \left(q_t^{(3)}(x) + q_t^{(3)} * \sum_{n=1}^{\infty} \frac{t^n \mu^{(*n)}}{n!}(x) \right), \quad x \in \mathbb{R}^3. \tag{61}$$

By (57) and the fact that $\operatorname{supp}(\mu) \subset B^c(0, 1)$ we get $\sup_{x \in B(0, 13/16)} |q_t^{(3)} * \mu(x)| \le M_1 < \infty$. Since $\gamma \in (1/2, 1)$ we get that the function $\mu \in L^2(\mathbb{R}^3)$ so $||\mu^{(*2)}||_\infty \le M_2 < \infty$. It follows that $||\mu^{(*n)}||_\infty \le M_2 M^{n-2}$, $n \ge 2$. Hence for $t \in (0, 1]$ and $x \in B(0, 1/2)$ we have $\left| q_t^{(3)} * \sum_{n=1}^{\infty} \frac{t^n \mu^{(*n)}}{n!}(x) \right| \le M_1 + M_2 + M_2 \sum_{n=2}^{\infty} \frac{t^n M^{n-2}}{n!} \le M_1 + M_2 + M_2 e^M$. This and (61) gives (56). $\square$

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