MID-CONCAVITY OF SURVIVAL PROBABILITY FOR ISOTROPIC LÉVY PROCESSES

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Abstract. Let X be a symmetric, pure jump, unimodal Lévy process in $\mathbb{R}$ with an infinite Lévy measure. We prove that for any fixed $t > 0$ the survival probability $P^x(\tau_{(-a,a)} > t)$ is nondecreasing on $(-a, 0]$, nonincreasing on $[0, a)$ and concave on $(-a/2, a/2)$, where $a > 0$ and $\tau_{(-a,a)}$ is the first exit time of the process X from $(-a, a)$. We also show a similar statement for sets $(-a, a) \times F \subset \mathbb{R}^d$.

1. Introduction

The main purpose of this paper is to investigate the monotonicity and concavity properties of the survival probability for some Lévy processes in $\mathbb{R}^d$. Let $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$ be the first exit time of an open, nonempty set $D \subset \mathbb{R}^d$ of the process X. We first formulate our result in one-dimensional setting.

Theorem 1.1. *Let X be a symmetric, pure jump, unimodal Lévy process in $\mathbb{R}$ with an infinite Lévy measure. Let $D = (-a, a)$, where $a > 0$. Put $\psi_t^D(x) = P^x(\tau_D > t)$ for $t \geq 0$ and $x \in \mathbb{R}$. Then for any $t > 0$ the function $x \to \psi_t^D(x)$ is nondecreasing on $(-a, 0]$, nonincreasing on $[0, a)$ and concave on $(-a/2, a/2)$.*

The next theorem is the generalization of the above result to higher dimensions.

Theorem 1.2. *Let X be an isotropic, pure jump, unimodal Lévy process in $\mathbb{R}^d$, $d \geq 2$ with an infinite Lévy measure. Let $D = (-a, a) \times F$, where $a > 0$ and $F \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain. Put $\psi_t^D(x) = P^x(\tau_D > t)$ for $t \geq 0$, $x \in \mathbb{R}^d$ and let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. Then for any $t > 0$ and $\tilde{x} \in \{0\} \times F$ the function $y \to \psi_t^D(ye_1 + \tilde{x})$ is nondecreasing on $(-a, 0]$, nonincreasing on $[0, a)$ and concave on $(-a/2, a/2)$.*

In Section 4 we will apply these results to obtain analogous properties of first eigenfunctions for the related Dirichlet eigenvalue problem.

Remark 1.3. The property that the function $x \to \psi_t^{(-a,a)}(x)$ or $y \to \psi_t^{(-a,a)\times F}(ye_1+\tilde{x})$ is concave on $(-a/2, a/2)$ is called *mid-concavity* (see Definition 1.1 in [2]).

The above results for isotropic α-stable processes in $\mathbb{R}^d$ (where $\alpha \in (0, 2]$) and intervals $(-a, a)$ or hyperrectangles $\prod_{i=1}^d(-a_i, a_i)$ are well known. They were proved by R. Bañuelos, T. Kulczycki and P. Méndez-Hernández in [2]. Indeed, the methods from [2] allow to extend these results for intervals or hyperrectangles to arbitrary subordinated Brownian motions in $\mathbb{R}^d$.

The main novelty of the results in this paper is that they concern arbitrary isotropic pure jump, unimodal Lévy processes in $\mathbb{R}^d$ with an infinite Lévy measure. The method used in the proof of Theorems 1.1, 1.2 is completely different

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than the method used in [2]. The key idea of the proof of Theorems 1.1, 1.2 is probabilistic. A very important step in this proof is the use of some results concerning the so-called *difference processes* which were introduced in [8].

The proof in [2] is analytical. The main idea in [2] (for $D = (-a, a)$) is to prove some properties of

$$\int_{-a}^{a} \dots \int_{-a}^{a} \prod_{i=1}^{n} p_{t_i}(x_{i-1} - x_i)\, dx_1 \dots dx_n$$

for gaussian kernels $p_t(x)$ and then use subordination to show monotonicity and midconcavity for

$$x \to P^x(X_{t_1} \in D, \dots, X_{t_n} \in D) \tag{1}$$

The results for $P^x(\tau_D > t)$ in [2] follows by a limiting procedure.

Note that in this paper we do not study properties of the function (1) but we study only properties of $P^x(\tau_D > t)$.

Very recently many researchers have been studying convexity properties of solutions of equations involving fractional Laplacians see [1], [4], [6], [7], [10]. In particular, concavity properties of the first eigenfunction for the Dirichlet eigenvalue problem on an interval for the fractional Laplacians have been studied in [1] and [6]. In this paper, using a probalistic approach, we obtain concavity properties of the first eigenfunction for the Dirichlet eigenvalue problem on an interval for much more general nonlocal operators, namely generators of the isotropic unimodal Lévy processes.

The paper is organized as follows. In Section 2 we present notation and collect some known facts needed in the rest of the paper. Section 3 contains proofs of Theorems 1.1, 1.2. In Section 4 we present regularity results of first eigenfunctions for the related Dirichlet eigenvalue problem.

2. Preliminaries

For $x \in \mathbb{R}^d$ and $r > 0$ we let $B(x, r) = \{y \in \mathbb{R}^d : |y - x| < r\}$. A Borel measure on $\mathbb{R}^d$ is called *isotropic unimodal* if on $\mathbb{R}^d \setminus \{0\}$ it is absolutely continuous with respect to the Lebesgue measure and has a finite radial, radially nonincreasing density function (such measures may have an atom at the origin).

A Lévy process $X = (X_t, t \ge 0)$ in $\mathbb{R}^d$ is called isotropic unimodal if its transition probability $p_t(dx)$ is isotropic unimodal for all $t > 0$. When additionally X is a pure-jump process then the following Lévy-Khintchine formula holds for $t > 0$ and $\xi \in \mathbb{R}^d$,

$$E^0 e^{i\xi X_t} = \int_{\mathbb{R}^d} e^{i\xi x} p_t(dx) = e^{-t\psi(\xi)} \quad \text{where} \quad \psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi x))\nu(dx).$$

ψ is the characteristic exponent of X and ν is the Lévy measure of X. E^0 is the expected value for the process X starting from 0. Recall that a Lévy measure is a measure concentrated on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$. Isotropic unimodal pure-jump Lévy measures are characterized in [12] by unimodal Lévy measures $\nu(dx) = \nu(x)\, dx = \nu(|x|)\, dx$.

Unless explicitly stated otherwise in what follows we assume that X is a pure-jump isotropic unimodal Lévy process in $\mathbb{R}^d$ with (isotropic unimodal) infinite Lévy measure ν. Then for any $t > 0$ the measure $p_t(dx)$ has a radial, radially nonincreasing density function $p_t(x) = p_t(|x|)$ on $\mathbb{R}^d$ with no atom at the origin. However, it may happen that $p_t(0) = \infty$, for some $t > 0$. As usual, we denote by P^x and E^x the probability measure and the corresponding expectation for the the process starting from $x \in \mathbb{R}^d$.

Let $D \subset \mathbb{R}^d$ be an open, nonempty set. We define a *killed process* X_t^D by $X_t^D = X_t$ if $t < \tau_D$ and $X_t^D = \partial$ otherwise, where ∂ is some point adjoined to D. The transition density for X_t^D on D is given by

$$p_D(t,x,y) = p_t(x-y) - E^x(p_{t-\tau_D}(X(\tau_D)-y),\, t > \tau_D), \quad x,y \in D,\, t > 0, \tag{2}$$

that is for any Borel set $A \subset \mathbb{R}^d$ we have

$$P^x(X_t^D \in A) = \int_A p_D(t,x,y)\,dy, \quad x \in D,\, t > 0.$$

We have $p_D(t,x,y) = p_D(t,y,x)$, $x,y \in D$, $t > 0$. We define the *Green function* for X_t^D by

$$G_D(x,y) = \int_0^\infty p_D(t,x,y)\,dt, \quad x,y \in D,$$

$G_D(x,y) = 0$ if $x \notin D$ or $y \notin D$.

Let $D \subset \mathbb{R}^d$ be an open, nonempty set. The distribution $P^x(X(\tau_D) \in \cdot)$ is called the *harmonic measure* with respect to X. The harmonic measure for Borel sets $A \subset (\overline{D})^c$ is given by the Ikeda-Watanabe formula [5],

$$P^x(X(\tau_D) \in A) = \int_A \int_D G_D(x,y)\nu(y-z)\,dy\,dz, \quad x \in D. \tag{3}$$

When $D \subset \mathbb{R}^d$ is a bounded, open Lipschitz set then we have [11], [9],

$$P^x(X(\tau_D) \in \partial D) = 0, \quad x \in D. \tag{4}$$

It follows that for such sets D the Ikeda-Watanabe formula (3) holds for any Borel set $A \subset D^c$. Let $D \subset \mathbb{R}^d$ be an open, nonempty set. For any $s > 0$, $x \in D$, $z \in (\overline{D})^c$ put

$$h_D(x,s,z) = \int_D p_D(s,x,y)\nu(y-z)\,dy. \tag{5}$$

By the Ikeda-Watanabe formula [5] for any Borel $A \subset (0,\infty)$, $B \subset (\bar{D})^c$ we have

$$P^x(\tau_D \in A, X(\tau_D) \in B) = \int_A \int_B h_D(x,s,z)\,dz\,ds, \quad x \in D. \tag{6}$$

If (4) holds then we can take $B \subset D^c$ in (6).

3. The monotonicity and midconcavity

We will prove both Theorems 1.1, 1.2 simultaneously. Let X be an isotropic, pure jump, unimodal Lévy process in $\mathbb{R}^d$, $d \geq 1$ with an infinite Lévy measure. Let $D = (-a,a) \times F$, where $a > 0$ and $F \subset \mathbb{R}^{d-1}$ is a bounded Lipschitz domain (or $D = (-a,a)$ when $d = 1$). Put $e_1 = (1,0\ldots,0) \in \mathbb{R}^d$. Note that for any $x \in D$ we have $P^x(X(\tau_D) \in \partial D) = 0$.

The key point in this section is the following result.

Proposition 3.1. *Let $U = (b,c) \times F$ (or $U = (b,c)$ when $d = 1$), where $-a \leq b < c \leq a$. Put $l(U) = b$, $r(U) = c$, $m(U) = (b+c)/2$, $U_- = (b, m(U)) \times F$, $U_+ = (m(U), c) \times F$, $H_-(U) = (-\infty, m(U)) \times \mathbb{R}^{d-1}$, $H_+(U) = (m(U), \infty) \times \mathbb{R}^{d-1}$*

(or $U_- = (b, m(U))$, $U_+ = (m(U), c)$, $H_-(U) = (-\infty, m(U))$, $H_+(U) = (m(U), \infty)$ when $d = 1$). For any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let

$$T_U(x) = x + 2e_1(m(U) - x_1),$$

(this is the reflection with respect to the hyperplane $x_1 = m(U)$, or with respect to a point $m(U)$ when $d = 1$). For any $s > 0$, $x \in U_+$, $z \in (\overline{U})^c$ put

$$f_s^U(x,z) = \int_{U_+} (p_U(s,x,y) - p_U(s,T_U(x),y))\,(\nu(y-z) - \nu(T_U(y)-z))\,dy.$$

For any $x \in U_+$, $t > 0$ we have

$$\psi_t^D(x) - \psi_t^D(T_U(x)) = \int_{U^c}\int_0^t f_s^U(x,z)\psi_{t-s}^D(z)\,ds\,dz, \tag{7}$$

$$f_s^U(x,z) \geq 0 \quad \text{for} \quad s > 0,\ z \in H_+(U) \setminus \overline{U_+}, \tag{8}$$

$$f_s^U(x,z) \leq 0 \quad \text{for} \quad s > 0,\ z \in H_-(U) \setminus \overline{U_-}. \tag{9}$$

Proof. By the strong Markov property and (6) for any $x \in U$, $t > 0$ we have

$$\begin{aligned}
\psi_t^D(x) &= P^x(\tau_D > t) \\
&= P^x(\tau_U > t) + E^x\left(\tau_U \leq t, \left[P^{X(\tau_U)}(\tau_D > t-s)\right]_{s=\tau_U}\right) \\
&= \psi_t^U(x) + \int_U\int_0^t p_U(s,x,y)\int_{U^c}\psi_{t-s}^D(z)\nu(y-z)\,dz\,ds\,dy.
\end{aligned}$$

It follows that

$$\psi_t^D(x) - \psi_t^D(T_U(x)) = \psi_t^U(x) - \psi_t^U(T_U(x)) \tag{10}$$

$$+\int_U\int_0^t (p_U(s,x,y) - p_U(s,T_U(x),y))\int_{U^c}\psi_{t-s}^D(z)\nu(y-z)\,dz\,ds\,dy. \tag{11}$$

For any $x \in U_+$, $t > 0$ by the symmetry of the process X and the definition of $T_U(x)$ we have

$$\psi_t^U(x) = \psi_t^U(T_U(x)). \tag{12}$$

For any $x \in U_+$, $t > 0$, $z \in (\overline{U})^c$ we also have

$$\begin{aligned}
&\int_U (p_U(s,x,y) - p_U(s,T_U(x),y))\,\nu(y-z)\,dy \\
=\ &\int_{U_+} (p_U(s,x,y) - p_U(s,T_U(x),y))\,\nu(y-z)\,dy \\
&+\int_{U_-} (p_U(s,x,y) - p_U(s,T_U(x),y))\,\nu(y-z)\,dy \\
=\ &\int_{U_+} (p_U(s,x,y) - p_U(s,T_U(x),y))\,\nu(y-z)\,dy \\
&+\int_{U_+} (p_U(s,x,T_U(y)) - p_U(s,T_U(x),T_U(y)))\,\nu\left(T_U(y)-z\right)\,dy \\
=\ &\int_{U_+} (p_U(s,x,y) - p_U(s,T_U(x),y))\,(\nu(y-z) - \nu\left(T_U(y)-z\right))\,dy,
\end{aligned}$$

(where in the last equality we used $p_U(s,T_U(x),T_U(y)) = p_U(s,x,y)$). Applying this, (10-11) and (12) we get (7).

Note that $p_U(s,x,y) - p_U(s,T_U(x),y)$ is the transition density of the so-called difference process (with respect to the hyperplane $x_1 = m(U)$ or the point $m(u)$ when $d = 1$) killed on exiting U_+ (see Section 4 in [8] for more details). By (19) in [8] and the first formula after the proof of Lemma 4.3 in [8] we obtain that $p_U(s,x,y) - p_U(s,T_U(x),y) \geq 0$ for any $s > 0$, $x, y \in U_+$. By unimodality of $\nu(x)$ we obtain that $\nu(y-z) - \nu(T_U(y) - z) \geq 0$ for any $y \in U_+$, $z \in H_+(U) \setminus \overline{U_+}$ and $\nu(y-z) - \nu(T_U(y) - z) \leq 0$ for any $y \in U_+$, $z \in H_-(U) \setminus \overline{U_-}$. This gives (8) and (9). □

Now we will show our main results.

proof of Theorems 1.1, 1.2. First we study monotonicity of ψ_t^D. Fix $\tilde{x} \in \{0\} \times F$ and $-a < x_1' < x_1'' \leq 0$. Put $x' = x_1' e_1 + \tilde{x}$, $x'' = x_1'' e_1 + \tilde{x}$ (or $x' = x_1'$, $x'' = x_1''$ when $d = 1$). Let $b = -a$, $x_* = (x_1' + x_1'')/2$, $c = -a + 2(x_* - (-a)) = a + x_1' + x_1''$, $U = (b,c) \times F$ (or $U = (b,c)$ when $d = 1$). Note that $m(U) = x_*$ and $T_U(x'') = x'$. By (7) for any $t > 0$ we get

$$\begin{aligned} \psi_t^D(x'') - \psi_t^D(x') &= \psi_t^D(x'') - \psi_t^D(T_U(x'')) && (13)\\ &= \int_{U^c} \int_0^t f_s^U(x'',z) \psi_{t-s}^D(z)\, ds\, dz && (14)\\ &= \int_{D \setminus \overline{U}} \int_0^t f_s^U(x'',z) \psi_{t-s}^D(z)\, ds\, dz. && (15) \end{aligned}$$

Note that $D \setminus \overline{U} = (c,a) \times F$ (or $D \setminus \overline{U} = (c,a)$ when $d = 1$) and $c = a + x_1' + x_1'' > m(U) = (x_1' + x_1'')/2$ so $D \setminus \overline{U} \subset H_+(U) \setminus \overline{U_+}$. This, (13-15) and (8) give $\psi_t^D(x'') \geq \psi_t^D(x')$. It follows that the function $y \to \psi_t^D(ye_1 + \tilde{x})$ (or $y \to \psi_t^D(y)$ when $d = 1$) is nondecreasing on $(-a, 0]$. By symmetry of the process X and the domain D the function $y \to \psi_t^D(ye_1 + \tilde{x})$ (or $y \to \psi_t^D(y)$ when $d = 1$) is nonincreasing on $[0, a)$.

Now we will study midconcavity of the function ψ_t^D. Fix $\tilde{x} \in \{0\} \times F$ and $-a/2 < x_1' < x_1'' < x_1''' \leq 0$ such that $x_1'' - x_1' = x_1''' - x_1''$. Put $x' = x_1' e_1 + \tilde{x}$, $x'' = x_1'' e_1 + \tilde{x}$, $x''' = x_1''' e_1 + \tilde{x}$ (or $x' = x_1'$, $x'' = x_1''$, $x''' = x_1'''$ when $d = 1$). As above, let $b = -a$, $x_* = (x_1' + x_1'')/2$, $c = -a + 2(x_* - (-a)) = a + x_1' + x_1''$, $U = (b,c) \times F$ (or $U = (b,c)$ when $d = 1$). We have $m(U) = x_*$ and $T_U(x'') = x'$. Note that $l(U) = -a$ and $r(U) = c = a + x_1' + x_1'' \in (0,a)$ (because $-a/2 < x_1' < x_1'' < 0$).

Let $v_1 = x_1'' - x_1'$ and $v = v_1 e_1$. Put

$$W = U + v.$$

Note that $W = (b + v_1, c + v_1) \times F$ (or $W = (b + v_1, c + v_1)$ when $d = 1$), $m(W) = m(U) + v_1$, $l(W) = l(U) + v_1$, $r(W) = r(U) + v_1 = a + 2x_1'' \in (0,a)$.

We have

$$T_W(x) = x + 2e_1(m(W) - x_1).$$

It follows that

$$\begin{aligned} T_W(x''') &= x''' + 2e_1(m(W) - x_1''')\\ &= x''' + 2e_1 \left(\frac{x_1''' + x_1''}{2} - x_1''' \right)\\ &= x''' + e_1(x_1'' - x_1''')\\ &= x''. \end{aligned}$$

Using this and Proposition 3.1 applied to W we get for any $t > 0$

$$\begin{aligned} \psi_t^D(x''') - \psi_t^D(x'') &= \psi_t^D(x''') - \psi_t^D(T_W(x''')) && (16) \\ &= \int_{W^c} \int_0^t f_s^W(x''', z) \psi_{t-s}^D(z) \, ds \, dz. && (17) \end{aligned}$$

Note that $W_+ = U_+ + v$ and $x''' = x'' + v$. Using this and the definition of $f_s^W(x, z)$ we get for any $s > 0$, $z \in W^c$

$$\begin{aligned} & f_s^W(x''', z) && (18) \\ = & \int_{W_+} (p_W(s, x''', y) - p_W(s, T_W(x'''), y)) \, (\nu(y - z) - \nu(T_W(y) - z)) \, dy && (19) \\ = & \int_{U_+ + v} (p_W(s, x'' + v, y) - p_W(s, T_W(x'' + v), y)) && (20) \\ & \qquad \times (\nu(y - z) - \nu(T_W(y) - z)) \, dy. && (21) \end{aligned}$$

Using substitution $q = y - v$ this is equal to

$$\begin{aligned} & \int_{U_+} (p_W(s, x'' + v, q + v) - p_W(s, T_W(x'' + v), q + v)) && (22) \\ & \qquad \times (\nu(q + v - z) - \nu(T_W(q + v) - z)) \, dq. && (23) \end{aligned}$$

For any $s > 0$, $q \in U_+$ we have

$$p_W(s, x'' + v, q + v) = p_{U+v}(s, x'' + v, q + v) = p_U(s, x'', q).$$

By the definition of T_W and the equality $m(W) = m(U) + v_1$ we get

$$T_W(x'' + v) = x'' + v + 2e_1(m(W) - x''_1 - v_1) = x'' + 2e_1(m(U) - x''_1) + v = T_U(x'') + v.$$

Hence for any $s > 0$, $q \in U_+$ we obtain

$$p_W(s, T_W(x'' + v), q + v) = p_{U+v}(s, T_U(x'') + v, q + v) = p_U(s, T_U(x''), q).$$

By similar arguments as above for any $q \in U_+$ we get $T_W(q + v) = T_U(q) + v$. Hence for any $q \in U_+$ and $z \in W^c$ we obtain

$$\nu(q + v - z) - \nu(T_W(q + v) - z) = \nu(q - (z - v)) - \nu(T_U(q) - (z - v)).$$

Using this, (18-21) and (22-23) we get for any $s > 0$, $z \in (\overline{W})^c$

$$\begin{aligned} & f_s^W(x''', z) \\ = & \int_{U_+} (p_U(s, x'', q) - p_U(s, T_U(x''), q)) \, (\nu(q - (z - v)) - \nu(T_U(q) - (z - v))) \, dq \\ = & f_s^U(x'', z - v). \end{aligned}$$

Using this, the fact that $W^c = U^c + v$ and (16-17) we get for any $t > 0$

$$\begin{aligned} \psi_t^D(x''') - \psi_t^D(x'') &= \int_{W^c} \int_0^t f_s^U(x'', z - v) \psi_{t-s}^D(z) \, ds \, dz && (24) \\ &= \int_{U^c} \int_0^t f_s^U(x'', z) \psi_{t-s}^D(z + v) \, ds \, dz. && (25) \end{aligned}$$

Put

$$L(U) = (-\infty, l(U)) \times F \subset H_-(U), \quad R(U) = (r(U), \infty) \times F \subset H_+(U),$$

(or $L(U) = (-\infty, l(U)) \subset H_-(U)$, $R(U) = (r(U), \infty) \subset H_+(U)$ when $d = 1$). By (13-14) and (24-25) we get for any $t > 0$

$$\begin{aligned} &\psi_t^D(x'') - \psi_t^D(x') \\ &= \int_{L(U)} \int_0^t f_s^U(x'', z)\psi_{t-s}^D(z)\, ds\, dz + \int_{R(U)} \int_0^t f_s^U(x'', z)\psi_{t-s}^D(z)\, ds\, dz \\ &= \mathrm{I} + \mathrm{II}, \end{aligned}$$

$$\begin{aligned} &\psi_t^D(x''') - \psi_t^D(x'') \\ &= \int_{L(U)} \int_0^t f_s^U(x'', z)\psi_{t-s}^D(z + v)\, ds\, dz + \int_{R(U)} \int_0^t f_s^U(x'', z)\psi_{t-s}^D(z + v)\, ds\, dz \\ &= \mathrm{III} + \mathrm{IV}. \end{aligned}$$

Since $l(U) = -a$ we get $\mathrm{I} = 0$. Since $L(U)+v \subset H_-(U)$ and by (9) $f_s^U(x'', z) \le 0$ for $z \in H_-(U)$ we get $\mathrm{III} \le 0$. Recall that $r(U) > 0$ so monotonicity of $y \to \psi_t^D(ye_1+\tilde{x})$ (or $y \to \psi_t^D(y)$ when $d = 1$) implies that for any $z \in R(U)$, $t > 0$, $s \in (0, t)$ we have $\psi_{t-s}^D(z + v) \le \psi_{t-s}^D(z)$ so $\mathrm{IV} \le \mathrm{II}$. Hence for any $t > 0$ we get

$$\psi_t^D(x'') - \psi_t^D(x') \ge \psi_t^D(x''') - \psi_t^D(x'').$$

Recall that $-a/2 < x_1' < x_1'' < x_1''' \le 0$, where $x_1'' - x_1' = x_1''' - x_1''$. Since x_1', x_1''' could be chosen arbitrarily we get that $y \to \psi_t^D(ye_1 + \tilde{x})$ (or $y \to \psi_t^D(y)$ when $d = 1$) is concave on $(-a/2, 0]$. By the symmetry we obtain that $y \to \psi_t^D(ye_1 + \tilde{x})$ (or $y \to \psi_t^D(y)$ when $d = 1$) is concave on $[0, a/2)$. □

4. The shape of the first eigenfunction

Let us recall that X is a pure-jump isotropic unimodal Lévy process in $\mathbb{R}^d$ with an infinite Lévy measure ν, ψ is the characteristic exponent of X and p_t is its transition density. In this section we additionally assume that

$$\lim_{|x|\to\infty} \frac{\psi(x)}{\log|x|} = \infty. \tag{26}$$

This guarantees that for any $t > 0$ the function p_t is continuous and bounded on $\mathbb{R}^d$.

Let $D \subset \mathbb{R}^d$ be a bounded open set. The condition (26) and formula (2) imply that for any fixed $t > 0$, $x \in D$ the function $y \to p_D(t, x, y)$ is continuous on D. Since $p_D(t, x, y) = p_D(t, y, x)$, $t > 0$, $x, y \in D$ we obtain that for any fixed $t > 0$, $y \in D$ the function $x \to p_D(t, x, y)$ is continuous on D. The transition operator P_t^D for the killed process X_t^D is defined by

$$P_t^D f(x) = \int_D p_D(t, x, y) f(y)\, dy, \quad x \in D,\, t > 0.$$

Now we introduce the Dirichlet eigenvalue problem on D for the Lévy process X. Such problem is well known in the literature see e.g. [3]. $\{P_t^D\}_{t\ge 0}$ forms a strongly continuous semigroup on $L^2(D)$. Since $p_D(t, x, y) \le p_t(x - y)$, $\|p_t\|_\infty < \infty$ and $D \subset \mathbb{R}^d$ is bounded we obtain that for any $t > 0$ the operator P_t^D is a Hilbert-Schmidt operator. From the general theory of semigroups there exists an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ in $L^2(D)$ and a corresponding sequence

$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \dots, \qquad \lim_{n\to\infty} \lambda_n = \infty,$$

such that for any $n \in \mathbb{N}$, $t > 0$, $x \in D$ we have

$$P_t^D \varphi_n(x) = e^{-\lambda_n t} \varphi_n(x). \tag{27}$$

λ_1 has multiplicity one and we may assume that $\varphi_1 > 0$ on D. By properties of $p_D(t,x,y)$ all eigenfunctions φ_n are bounded and continuous on D. It is well known that

$$p_D(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y), \quad t > 0,\ x, y \in D.$$

It follows that for any $t > 0$ and $x \in D$ we have

$$\begin{aligned} P^x(\tau_D > t) &= \int_D p_D(t,x,y)\, dy \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \int_D \varphi_n(y)\, dy. \end{aligned}$$

Hence for any $x \in D$ we have

$$\lim_{t \to \infty} e^{\lambda_1 t} P^x(\tau_D > t) = \varphi_1(x) \int_D \varphi_1(y)\, dy.$$

Using this and Theorems 1.1, 1.2 we immediately obtain the following results.

Corollary 4.1. *Let X be a symmetric, pure jump, unimodal Lévy process in $\mathbb{R}$ satisfying (26) with an infinite Lévy measure. Let $D = (-a, a)$, where $a > 0$. Let φ_1 be the first eigenfunction of the spectral problem (27) on D for the process X. Then for any $t > 0$ the function $x \to \varphi_1(x)$ is nondecreasing on $(-a, 0]$, nonincreasing on $[0, a)$ and concave on $(-a/2, a/2)$.*

Corollary 4.2. *Let X be an isotropic, pure jump, unimodal Lévy process in $\mathbb{R}^d$, $d \geq 2$ satisfying (26) with an infinite Lévy measure. Let $D = (-a, a) \times F$, where $a > 0$ and $F \subset \mathbb{R}^{d-1}$ be a bounded Lipschitz domain. Let φ_1 be the first eigenfunction of the spectral problem (27) on D for the process X. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$. Then for any $t > 0$ and $\tilde{x} \in \{0\} \times F$ the function $y \to \varphi_1(ye_1 + \tilde{x})$ is nondecreasing on $(-a, 0]$, nonincreasing on $[0, a)$ and concave on $(-a/2, a/2)$.*

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