Analiza Matematyczna

Przykłady:

Funkcje wielu zmiennych: pochodne cząstkowe.

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Przykład **14.1**:

Korzystając z definicji oblicz wszystkie pochodne cząstkowe pierwszego rzędu funkcji $f(x,y) = xy + x^2 + y - 2x$ w punkcie $(1,0)$:

- $\frac{\partial f}{\partial x}(1,0) = \lim\limits_{\Delta x \to 0} \frac{f(1+\Delta x, 0) - f(1,0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \frac{(1+\Delta x)^2 - 2(1+\Delta x) - (-1)}{\Delta x} =$
$= \lim\limits_{\Delta x \to 0} \frac{1 + 2\Delta x + \Delta x^2 - 2 - 2\Delta x + 1}{\Delta x} = \lim\limits_{\Delta x \to 0} \Delta x = 0$
- $\frac{\partial f}{\partial y}(1,0) = \lim\limits_{\Delta y \to 0} \frac{f(1, 0+\Delta y) - f(1,0)}{\Delta y} = \lim\limits_{\Delta y \to 0} \frac{2\Delta y - 1 - (-1)}{\Delta y} = \lim\limits_{\Delta y \to 0} 2 = 2$

Przykłady **14.2**:

Oblicz wszystkie pochodne cząstkowe pierwszego rzędu podanych funkcji:

(a) $f(x,y) = xy + x^2 + y - 2x$

- $D_f = \mathbb{R}^2$
- $\frac{\partial f}{\partial x}(x,y) = y + 2x - 2$
- $\frac{\partial f}{\partial y}(x,y) = x + 1$

(b) $f(x,y) = \frac{e^x}{\ln(x+y)}$

- $D_f : x + y > 0, x + y \neq 1$
- $\frac{\partial f}{\partial x}(x,y) = \frac{e^x \ln(x+y) - e^x \cdot \frac{1}{x+y} \cdot 1}{\ln^2(x+y)}$
- $\frac{\partial f}{\partial y}(x,y) = e^x \left(\frac{-1}{\ln^2(x+y)} \right) \cdot \frac{1}{x+y} \cdot 1$

(c) $f(x,y) = \sin^2(x - y^2)$

- $D_f = \mathbb{R}^2$
- $\frac{\partial f}{\partial x}(x,y) = 2\sin(x-y^2)\cos(x-y^2) \cdot 1$
- $\frac{\partial f}{\partial y}(x,y) = 2\sin(x-y^2)\cos(x-y^2) \cdot (-2y)$

(d) $f(x,y) = x^y$

- $D_f : x > 0$
- $\frac{\partial f}{\partial x}(x,y) = yx^{y-1}$
- $\frac{\partial f}{\partial y}(x,y) = x^y \ln x$

(e) $f(x,y,z) = y - \sqrt{x^2 + z^3}$

- $D_f : x^2 + z^3 \geqslant 0$
- $\dfrac{\partial f}{\partial x}(x,y,z) = -\dfrac{1}{2}(x^2+z^3)^{-1/2} \cdot 2x$ dla $x^2 + z^3 > 0$
- $\dfrac{\partial f}{\partial y}(x,y,z) = 1$
- $\dfrac{\partial f}{\partial z}(x,y,z) = -\dfrac{1}{2}(x^2+z^3)^{-1/2} \cdot 3z^2$ dla $x^2 + z^3 > 0$

(f) $f(x,y,z) = \sqrt[3]{\operatorname{arctg}(x + e^{yz})}$

- $D_f = \mathbb{R}^2$
- $\dfrac{\partial f}{\partial x}(x,y,z) = \dfrac{1}{3}(\operatorname{arctg}(x+e^{yz}))^{-2/3} \cdot \dfrac{1}{(x+e^{yz})^2+1} \cdot 1$
- $\dfrac{\partial f}{\partial y}(x,y,z) = \dfrac{1}{3}(\operatorname{arctg}(x+e^{yz}))^{-2/3} \cdot \dfrac{1}{(x+e^{yz})^2+1} \cdot e^{yz} \cdot z$
- $\dfrac{\partial f}{\partial z}(x,y,z) = \dfrac{1}{3}(\operatorname{arctg}(x+e^{yz}))^{-2/3} \cdot \dfrac{1}{(x+e^{yz})^2+1} \cdot e^{yz} \cdot y$
  dla wszystkich pochodnych $x + e^{yz} \neq 0$

Przykłady **14.3**:

Oblicz wszystkie pochodne cząstkowe drugiego rzędu podanych funkcji i sprawdź, czy pochodne cząstkowe mieszane są równe:

(a) $f(x,y) = \ln(x-y)$

- $D_f : x - y > 0$
- $\dfrac{\partial f}{\partial x}(x,y) = \dfrac{1}{x-y}, \qquad \dfrac{\partial f}{\partial y}(x,y) = \dfrac{1}{x-y} \cdot (-1) = \dfrac{1}{y-x}$
- $\dfrac{\partial^2 f}{\partial x^2}(x,y) = \dfrac{\partial}{\partial x}\left(\dfrac{1}{x-y}\right) = -\dfrac{1}{(x-y)^2}$

  $\dfrac{\partial^2 f}{\partial y \partial x}(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{1}{x-y}\right) = -\dfrac{1}{(x-y)^2} \cdot (-1)$

  $\dfrac{\partial^2 f}{\partial x \partial y}(x,y) = \dfrac{\partial}{\partial x}\left(\dfrac{1}{y-x}\right) = -\dfrac{1}{(y-x)^2} \cdot (-1)$

  $\dfrac{\partial^2 f}{\partial y^2}(x,y) = \dfrac{\partial}{\partial y}\left(\dfrac{1}{y-x}\right) = -\dfrac{1}{(y-x)^2}$
- Sprawdzenie: $\dfrac{\partial^2 f}{\partial y \partial x}(x,y) = \dfrac{1}{(x-y)^2} = \dfrac{\partial^2 f}{\partial x \partial y}(x,y)$

(b) $f(x,y) = e^{\frac{x}{y}}$

- $D_f : y \neq 0$
- $\frac{\partial f}{\partial x}(x,y) = \frac{1}{y}e^{\frac{x}{y}}, \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{x}{y^2}e^{\frac{x}{y}}$
- $\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{1}{y^2}e^{\frac{x}{y}}$

  $\frac{\partial^2 f}{\partial y \partial x}(x,y) = -\frac{1}{y^2}e^{\frac{x}{y}} + \frac{1}{y}\left(-\frac{x}{y^2}e^{\frac{x}{y}}\right)$

  $\frac{\partial^2 f}{\partial x \partial y}(x,y) = -\frac{1}{y^2}e^{\frac{x}{y}} - \frac{x}{y^2} \cdot \frac{1}{y}e^{\frac{x}{y}}$

  $\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2x}{y^3}e^{\frac{x}{y}} - \frac{x}{y^2}\left(-\frac{x}{y^2}e^{\frac{x}{y}}\right) = \frac{x(x+2y)}{y^4}e^{\frac{x}{y}}$
- Sprawdzenie: $\frac{\partial^2 f}{\partial y \partial x}(x,y) = -\frac{x+y}{y^3}e^{\frac{x}{y}} = \frac{\partial^2 f}{\partial x \partial y}(x,y)$

(c) $f(x,y,z) = x^2 + y^3x - 2x^3y^2z^5$

- $D_f = \mathbb{R}^2$
- $\frac{\partial f}{\partial x}(x,y,z) = 2x + y^3 - 6x^2y^2z^5, \quad \frac{\partial f}{\partial y}(x,y,z) = 3y^2x - 4x^3yz^5, \quad \frac{\partial f}{\partial z}(x,y,z) = -10x^3y^2z^4$
- $\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2 - 12xy^2z^5 \qquad \frac{\partial^2 f}{\partial x \partial y}(x,y,z) = 3y^2 - 12x^2yz^5$

  $\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = 3y^2 - 12x^2yz^5 \qquad \frac{\partial^2 f}{\partial y^2}(x,y,z) = 6yx - 4x^3z^5$

  $\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = -30x^2y^2z^4 \qquad \frac{\partial^2 f}{\partial z \partial y}(x,y,z) = -20x^3yz^4$

  $$\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = -30x^2y^2z^4$$
  $$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = -20x^3yz^4$$
  $$\frac{\partial^2 f}{\partial z^2}(x,y,z) = -40x^3y^2z^3$$

- Sprawdzenie: $\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = 3y^2 - 12x^2yz^5 = \frac{\partial^2 f}{\partial x \partial y}(x,y,z)$

  $\frac{\partial^2 f}{\partial z \partial x}(x,y,z) = -30x^2y^2z^4 = \frac{\partial^2 f}{\partial x \partial z}(x,y,z)$

  $\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = -20x^3yz^4 = \frac{\partial^2 f}{\partial z \partial y}(x,y,z)$

Przykłady **14.4**:

Oblicz wskazane pochodne cząstkowe podanych funkcji:

(a) $\dfrac{\partial^3 f}{\partial x \partial y^2}(x,y)$ dla $f(x,y)=\cos\left(\dfrac{y}{x}\right)$

- $D_f : x \neq 0$
- Mamy obliczyć $\dfrac{\partial^3 f}{\partial x \partial y^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)\right)$
- $\dfrac{\partial f}{\partial y}(x,y) = -\sin\left(\dfrac{y}{x}\right)\cdot\dfrac{1}{x}$
- $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)(x,y) = \dfrac{\partial}{\partial y}\left(-\dfrac{1}{x}\sin\left(\dfrac{y}{x}\right)\right) = -\dfrac{1}{x}\cos\left(\dfrac{y}{x}\right)\cdot\dfrac{1}{x} = -\dfrac{1}{x^2}\cos\left(\dfrac{y}{x}\right)$
- $\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)\right)(x,y) = \dfrac{\partial}{\partial x}\left(-\dfrac{1}{x^2}\cos\left(\dfrac{y}{x}\right)\right) = \dfrac{2}{x^3}\cos\left(\dfrac{y}{x}\right) - \dfrac{1}{x^2}\left(-\sin\left(\dfrac{y}{x}\right)\right)\cdot\left(-\dfrac{y}{x^2}\right)$

Odp.: $\dfrac{\partial^3 f}{\partial x \partial y^2}(x,y) = \dfrac{2}{x^3}\cos\left(\dfrac{y}{x}\right) - \dfrac{y}{x^4}\sin\left(\dfrac{y}{x}\right)$

(b) $\dfrac{\partial^5 f}{\partial x^2 \partial z \partial y^2}(x,y,z)$ dla $f(x,y,z)=x^2y^3z^4$

- $D_f = \mathbb{R}^3$
- Mamy obliczyć $\dfrac{\partial^5 f}{\partial x^2 \partial z \partial y^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial z}\left(\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)\right)\right)\right)$
- $\dfrac{\partial f}{\partial y}(x,y,z) = 3x^2y^2z^4$
- $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)(x,y,z) = \dfrac{\partial}{\partial y}\left(3x^2y^2z^4\right) = 6x^2yz^4$
- $\dfrac{\partial}{\partial z}\left(\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)\right)(x,y,z) = \dfrac{\partial}{\partial z}\left(6x^2yz^4\right) = 24x^2yz^3$
- $\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial z}\left(\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)\right)\right)(x,y,z) = \dfrac{\partial}{\partial x}\left(24x^2yz^3\right) = 48xyz^3$
- $\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial z}\left(\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right)\right)\right)\right)(x,y,z) = \dfrac{\partial}{\partial x}\left(48xyz^3\right) = 48yz^3$

Odp.: $\dfrac{\partial^5 f}{\partial x^2 \partial z \partial y^2}(x,y,z) = 48yz^3$