# Analiza Matematyczna

Przykłady: **Twierdzenie de l'Hospitala.**

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Przykład **5.1**:

Stosując regułę de l'Hospitala oblicz podane granice:

(a) $\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\left[\dfrac{0}{0}\right]\overset{H}{=}\lim\limits_{x\to 0}\dfrac{(x-\sin x)'}{(x^3)'}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{3x^2}=\left[\dfrac{0}{0}\right]\overset{H}{=}\lim\limits_{x\to 0}\dfrac{\sin x}{6x}=\dfrac{1}{6}\cdot 1=\dfrac{1}{6}$

(b) $\lim\limits_{x\to\infty}\dfrac{\pi-2\mathrm{arctg}x}{\ln\left(1+\frac{1}{x^2}\right)}=\left[\dfrac{0}{0}\right]\overset{H}{=}\lim\limits_{x\to\infty}\dfrac{\frac{-2}{1+x^2}}{\frac{1}{1+\frac{1}{x^2}}\cdot\left(\frac{-2}{x^3}\right)}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{1+x^2}}{\frac{1}{1+x^2}\cdot\frac{1}{x}}=\lim\limits_{x\to\infty}x=\infty$

(c) $\lim\limits_{x\to\infty}\dfrac{x+\ln x}{x\ln x}=\left[\dfrac{\infty}{\infty}\right]\overset{H}{=}\lim\limits_{x\to\infty}\dfrac{1+\frac{1}{x}}{\ln x+x\cdot\frac{1}{x}}=\lim\limits_{x\to\infty}\dfrac{1+\frac{1}{x}}{\ln x+1}=\dfrac{1+0}{\infty+1}=\dfrac{1}{\infty}=0$

(d) $\lim\limits_{x\to 0+}\dfrac{\mathrm{ctg}\,x}{\ln x}=\left[\dfrac{-\infty}{-\infty}\right]\overset{H}{=}\lim\limits_{x\to 0+}\dfrac{-\frac{1}{\sin^2 x}}{\frac{1}{x}}=\lim\limits_{x\to 0+}\left(-\dfrac{1}{x}\right)\cdot\left(\dfrac{x}{\sin x}\right)^2=-\dfrac{1}{0+}\cdot 1^2=-\infty$

(e) $\lim\limits_{x\to 0+}x^2\ln x=[0\cdot(-\infty)]=\lim\limits_{x\to 0+}\dfrac{\ln x}{\left(\frac{1}{x^2}\right)}=\left[\dfrac{-\infty}{\infty}\right]\overset{H}{=}\lim\limits_{x\to 0+}\dfrac{\frac{1}{x}}{-\frac{2}{x^3}}=\lim\limits_{x\to 0+}\left(-\dfrac{x^2}{2}\right)=0$

(Skorzystaliśmy z przekształacenia $f\cdot g=\dfrac{f}{\left(\frac{1}{g}\right)}$.)

(f) $\lim\limits_{x\to\infty}(e^x-x)=[\infty-\infty]=\lim\limits_{x\to\infty}e^x\left(1-\frac{x}{e^x}\right)=\infty(1-0)=\infty$

- **Obliczenie pomocnicze:** $\lim\limits_{x\to\infty}\frac{x}{e^x}=\left[\dfrac{-\infty}{\infty}\right]\overset{H}{=}\lim\limits_{x\to\infty}\frac{1}{e^x}=\dfrac{1}{\infty}=0$

(Wykorzystaliśmy tutaj niestandardowy pomysł własny.)

(g) $\lim\limits_{x\to\pi-}\left(\dfrac{1}{\sin x}-\dfrac{1}{\pi-x}\right)=[\infty-\infty]=\lim\limits_{x\to\pi-}\dfrac{\pi-x-\sin x}{(\pi-x)\sin x}=\left[\dfrac{0}{0}\right]\overset{H}{=}\lim\limits_{x\to\pi-}\dfrac{-1-\cos x}{-\sin x+(\pi-x)\cos x}=$

$=\left[\dfrac{0}{0}\right]\overset{H}{=}\lim\limits_{x\to\pi-}\dfrac{\sin x}{-\cos x-\cos x+(\pi-x)(-\sin x)}=\dfrac{0}{1+1+0}=0$

(Sprowadziliśmy do wspólnego mianownika.)

(h) $\lim\limits_{x\to 0}\left(\operatorname{ctg}^2 x - \dfrac{1}{x^2}\right) = [\infty - \infty] = \lim\limits_{x\to 0}\dfrac{x^2 - \frac{1}{\operatorname{ctg}^2 x}}{x^2\cdot\frac{1}{\operatorname{ctg}^2 x}} = \lim\limits_{x\to 0}\dfrac{x^2 - \operatorname{tg}^2 x}{x^2\operatorname{tg}^2 x} = \left[\dfrac{0}{0}\right] \overset{H}{=}$

$= \lim\limits_{x\to 0}\dfrac{2x - 2\operatorname{tg} x(1+\operatorname{tg}^2 x)}{2x\operatorname{tg}^2 x + x^2\cdot 2\operatorname{tg} x(1+\operatorname{tg}^2 x)} = \left[\dfrac{0}{0}\right] \overset{H}{=}$

$= \lim\limits_{x\to 0}\dfrac{1-(1+\operatorname{tg}^2 x)^2 - \operatorname{tg} x\cdot 2\operatorname{tg} x(1+\operatorname{tg}^2 x)}{\operatorname{tg}^2 x + x\cdot 2\operatorname{tg} x(1+\operatorname{tg}^2 x) + 2x\operatorname{tg} x(1+\operatorname{tg}^2 x) + x^2((1+\operatorname{tg}^2 x)^2 + 2\operatorname{tg}^2 x(1+\operatorname{tg}^2 x))} =$

$= \lim\limits_{x\to 0}\dfrac{1-1-2\operatorname{tg}^2 x - \operatorname{tg}^4 x - 2\operatorname{tg}^2 x(1+\operatorname{tg}^2 x)}{\operatorname{tg}^2 x + 4x\operatorname{tg} x(1+\operatorname{tg}^2 x) + x^2(1+3\operatorname{tg}^2 x)(1+\operatorname{tg}^2 x)} =$

$= \lim\limits_{x\to 0}\dfrac{-2\left(\frac{\operatorname{tg} x}{x}\right)^2 - \operatorname{tg}^2 x\left(\frac{\operatorname{tg} x}{x}\right)^2 - 2\left(\frac{\operatorname{tg} x}{x}\right)^2(1+\operatorname{tg}^2 x)}{\left(\frac{\operatorname{tg} x}{x}\right)^2 + 4\left(\frac{\operatorname{tg} x}{x}\right)(1+\operatorname{tg}^2 x) + (1+3\operatorname{tg}^2 x)(1+\operatorname{tg}^2 x)} = \dfrac{-2-0-2}{1+4+1} = -\dfrac{2}{3}$

(Skorzystaliśmy ze wzoru $f - g = \dfrac{\left(\frac{1}{g} - \frac{1}{f}\right)}{\left(\frac{1}{f\cdot g}\right)}$.)

(i) $\lim\limits_{x\to 0+} x^{\sin^2 x} = [0^0] = \lim\limits_{x\to 0+} e^{\sin^2 x\ln x} = e^{\lim\limits_{x\to 0+}\sin^2 x\ln x} = e^0 = 1$

- **Obliczenie pomocnicze:** $\lim\limits_{x\to 0+}\sin^2 x\ln x = \lim\limits_{x\to 0+}\left(\dfrac{\sin x}{x}\right)^2\cdot(x^2\ln x) = 1^2\cdot 0 = 0$
  na podstawie przykładu (e)

(j) $\lim\limits_{x\to\infty}\left(\frac{2}{\pi}\operatorname{arctg} x\right)^x = [1^\infty] = \lim\limits_{x\to\infty} e^{x\ln\left(\frac{2}{\pi}\operatorname{arctg} x\right)} = e^{\lim\limits_{x\to\infty} x\ln\left(\frac{2}{\pi}\operatorname{arctg} x\right)} = e^{-2/\pi}$

- **Obliczenie pomocnicze:** $\lim\limits_{x\to\infty} x\ln\left(\frac{2}{\pi}\operatorname{arctg} x\right) = [\infty\cdot 0] = \lim\limits_{x\to\infty}\dfrac{\ln\left(\frac{2}{\pi}\operatorname{arctg} x\right)}{\left(\frac{1}{x}\right)} = \left[\dfrac{0}{0}\right] \overset{H}{=}$

  $= \lim\limits_{x\to\infty}\dfrac{\frac{1}{\frac{2}{\pi}\operatorname{arctg} x}\cdot\frac{2}{\pi}\cdot\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty}\left(\dfrac{-1}{\operatorname{arctg} x}\right)\cdot\left(\dfrac{1}{\frac{1}{x^2}+1}\right) = \dfrac{-1}{\pi/2}\cdot\dfrac{1}{0+1} = -\dfrac{2}{\pi}$