Analiza Matematyczna

Przykłady: **Całki nieoznaczone.**

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Przykłady **8.1**:

Oblicz podane całki nieoznaczone.

(a) $\int (x-2e^x)dx = \int x dx - 2\int e^x dx = \dfrac{x^2}{2} - 2e^x + C,\ \ C\in\mathbb{R}$

(b) $\displaystyle\int \sin^2\left(\frac{x}{2}\right)dx = \left[\begin{array}{l}\cos x = 1-2\sin^2\dfrac{x}{2}\\ \sin^2\dfrac{x}{2} = \dfrac{1}{2}(1-\cos x)\end{array}\right] = \int \frac{1}{2}(1-\cos x)dx = \frac{1}{2}\left(\int 1dx - \int \cos x dx\right) =$

$= \dfrac{1}{2}(x-\sin x) + C,\ \ C\in\mathbb{R}$

(c) $\displaystyle\int \frac{x^2}{1+x^2}dx = \left[\frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = 1-\frac{1}{1+x^2}\right] = \int\left(1-\frac{1}{1+x^2}\right)dx = \int 1dx - \int\frac{1}{1+x^2}dx =$

$= x - \mathrm{arctg}x + C,\ \ C\in\mathbb{R}$

Przykłady **8.2**:

Korzystając z twierdzenia o całkowaniu przez części oblicz całki nieoznaczone:

(a) $\int x\sin x dx = \left[\begin{array}{ll} f=x & g'=\sin x\\ f'=1 & g=\int \sin x dx = -\cos x\end{array}\right] = -x\cos x - \int 1\cdot(-\cos x)dx =$

$= -x\cos x + \int \cos x dx = -x\cos x + \sin x + C,\ \ C\in\mathbb{R}$

(b) $\displaystyle\int x\mathrm{arctg}x dx = \left[\begin{array}{ll} f=\mathrm{arctg}x & g'=x\\ f'=\dfrac{1}{1+x^2} & g=\int x dx = \dfrac{x^2}{2}\end{array}\right] = \frac{x^2}{2}\mathrm{arctg}x - \int\frac{1}{1+x^2}\cdot\frac{x^2}{2}dx =$

$\displaystyle = \frac{x^2}{2}\mathrm{arctg}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx = \frac{x^2}{2}\mathrm{arctg}x - \frac{1}{2}(x-\mathrm{arctg}x) + C,\ \ C\in\mathbb{R}$

(na podstawie przykładu 8.1 (c))

(c) $\displaystyle\int \ln x dx = \left[\begin{array}{ll} f=\ln x & g'=1\\ f'=\dfrac{1}{x} & g=\int 1dx = x\end{array}\right] = x\ln x - \int\frac{1}{x}\cdot x dx =$

$= x\ln x - \int 1dx = x\ln x - x + C,\ \ C\in\mathbb{R}$

(d) $A = \int e^x\cos x dx = \left[\begin{array}{ll} f=\cos x & g'=e^x\\ f'=-\sin x & g=\int e^x dx = e^x\end{array}\right] = e^x\cos x + \int e^x\sin x dx =$

$= \left[\begin{array}{ll} f=\sin x & g'=e^x\\ f'=\cos x & g=e^x\end{array}\right] = e^x\cos x + (e^x\sin x - \int e^x\cos x dx) = e^x(\cos x+\sin x) - A + c,\ \ c\in\mathbb{R}$

Zatem $A = \int e^x\cos x dx = \dfrac{e^x(\cos x+\sin x)}{2} + C,\ \ C = c/2\in\mathbb{R}$

Przykłady **8.3**:

Stosując odpowiednie podstawienia oblicz podane całki nieoznaczone:

(a) $\int \sin^5 x \cos x dx = \left[\begin{array}{l} y = \sin x \\ dy = (\sin x)' dx = \cos x dx \end{array} \right] = \int y^5 dy = \frac{y^6}{6} = \frac{\sin^6 x}{6} + C, \;\; C \in \mathbb{R}$

(b) $\int \frac{dx}{x \ln^2 x} = \int \frac{1}{\ln^2 x} \cdot \frac{1}{x} dx = \left[\begin{array}{l} y = \ln x \\ dy = (\ln x)' dx = \frac{1}{x} dx \end{array} \right] = \int y^{-2} dy = \frac{y^{-1}}{-1} = \frac{-1}{\ln x} + C, \;\; C \in \mathbb{R}$

(c) $\int \frac{x^7 dx}{\sqrt{1-x^{16}}} = \int \frac{1}{\sqrt{1-(x^8)^2}} x^7 dx = \left[\begin{array}{l} y = x^8 \\ dy = (x^8)' dx = 8x^7 dx \end{array} \right] = \int \frac{1}{\sqrt{1-y^2}} \frac{1}{8} dy = \frac{1}{8} \arcsin y =$

$= \frac{1}{8} \arcsin(x^8) + C, \;\; C \in \mathbb{R}$

(d) $\int \frac{dx}{1+e^{3x}} = \left[\begin{array}{l} y = e^{3x} > 0 \\ x = \frac{1}{3} \ln y \\ dx = (\frac{1}{3} \ln y)' dy = \frac{1}{3y} dy \end{array} \right] = \int \frac{1}{1+y} \cdot \frac{1}{3y} dy = \frac{1}{3} \int \frac{1}{y(y+1)} dy =$

$= \left[\begin{array}{l} \frac{1}{y(y+1)} = \frac{a}{y} + \frac{b}{y+1} \\ 1 = a(y+1) + by \\ y = 0 \Rightarrow a = 1 \\ y = -1 \Rightarrow b = -1 \end{array} \right] = \frac{1}{3} \int \left(\frac{1}{y} - \frac{1}{y+1} \right) dy =$

$= \frac{1}{3} (\ln|y| - \ln|y+1|) = \frac{1}{3} (\ln e^{3x} - \ln(e^{3x}+1)) = x - \frac{1}{3} \ln(e^{3x}+1) + C, \;\; C \in \mathbb{R}$

Przykłady **8.4**:

Oblicz podane całki nieoznaczone z funkcji wymiernych:

(a) $\int \frac{1}{y(y+1)} dy$ obliczaliśmy w 7.3(d)

(b) $\int \frac{2x+4}{x^3-2x^2} dx = \left[\begin{array}{l} \frac{2x+4}{x^3-2x^2} = \frac{2x+4}{x^2(x-2)} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x-2} \\ 2x+4 = ax(x-2) + b(x-2) + cx^2 = (a+c)x^2 + (-2a+b)x - 2b \\ \left\{ \begin{array}{rcl} a+c & = & 0 \\ -2a+b & = & 2 \\ -2b & = & 4 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcr} a & = & -2 \\ b & = & -2 \\ c & = & 2 \end{array} \right. \end{array} \right] =$

$= \int \left(\frac{-2}{x} + \frac{-2}{x^2} + \frac{2}{x-2} \right) dx = -2 \int \frac{dx}{x} - 2 \int \frac{dx}{x^2} + 2 \int \frac{dx}{x-2} =$

$= -2 \ln|x| - 2 \left(\frac{-1}{x} \right) + 2 \ln|x-2| + C, \;\; C \in \mathbb{R}$

(c) $\int \frac{dx}{x^2-x+1} = \left[\begin{array}{l} \Delta = 1 - 4 < 0 \\ x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} = \frac{3}{4} \left(\left(\frac{2}{\sqrt{3}} \left(x - \frac{1}{2}\right) \right)^2 + 1 \right) = \frac{3}{4} \left(\left(\frac{2x-1}{\sqrt{3}} \right)^2 + 1 \right) \end{array} \right] =$

$= \frac{4}{3} \int \frac{dx}{\left(\frac{2x-1}{\sqrt{3}}\right)^2 + 1} = \left[\begin{array}{l} y = \frac{2x-1}{\sqrt{3}} \\ x = \frac{\sqrt{3}y+1}{2} \\ dx = \frac{\sqrt{3}}{2} dy \end{array} \right] = \frac{4}{3} \int \frac{\frac{\sqrt{3}}{2} dy}{y^2+1} = \frac{2\sqrt{3}}{3} \text{arctg} y = \frac{2\sqrt{3}}{3} \text{arctg} \left(\frac{2x-1}{\sqrt{3}} \right) + C,$

$C \in \mathbb{R}$

(d) $$\int \frac{2x+3}{x^2+2x+2}dx = \left[\begin{array}{l} \Delta = 4-8<0 \\ x^2+2x+2 = (x+1)^2+1 \\ (x^2+2x+2)' = 2x+2 \end{array} \right] =$$

$$= \int \frac{2x+2+1}{x^2+2x+2}dx = \int \frac{2x+2}{x^2+2x+2}dx + \int \frac{1}{(x+1)^2+1}dx = \ln(x^2+2x+2) + \text{arctg}(x+1) + C,$$
$C \in \mathbb{R}$

Obliczenia pomocnicze:

$$\int \frac{2x+2}{x^2+2x+2}dx = \left[\begin{array}{l} y = x^2+2x+2 \\ dy = (2x+2)dx \end{array} \right] = \int \frac{dy}{y} = \ln|y| = \ln(x^2+2x+2)$$

$$\int \frac{1}{(x+1)^2+1}dx = \left[\begin{array}{l} y = x+1 \\ dy = dx \end{array} \right] = \int \frac{dy}{y^2+1} = \text{arctg}y = \text{arctg}(x+1)$$

Przykłady **8.5**:

Oblicz podane całki nieoznaczone z funkcji trygonometrycznych:

(a) $$\int \frac{\sin^4 x}{\cos x}dx = \left[\begin{array}{l} R(u,v) = \frac{u^4}{v} \\ R(u,-v) = -R(u,v) \end{array} \middle| \; y = \sin x \right] = \int \frac{y^4}{\sqrt{1-y^2}} \cdot \frac{dy}{\sqrt{1-y^2}} =$$

$$= \int \frac{y^4}{1-y^2}dy = -\int \frac{y^4}{y^2-1}dy = \left[\begin{array}{l} y^4 = (y^2-1)(y^2+1)+1 \\ \frac{y^4}{y^2-1} = y^2+1+\frac{1}{(y-1)(y+1)} = y^2+1+\frac{a}{y-1}+\frac{b}{y+1} \\ 1 = a(y+1)+b(y-1) = (a+b)y+(a-b) \\ \left\{ \begin{array}{lcl} a+b &=& 0 \\ a-b &=& 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{lcl} a &=& 1/2 \\ b &=& -1/2 \end{array} \right. \end{array} \right] =$$

$$= -\int \left(y^2+1+\frac{1/2}{y-1}+\frac{-1/2}{y+1} \right) dy = -\left(\frac{y^3}{3}+y+\frac{1}{2}\ln|y-1|-\frac{1}{2}\ln|y+1| \right) =$$

$$= -\left(\frac{\sin^3 x}{3}+\sin x+\frac{1}{2}\ln|\sin x-1|-\frac{1}{2}\ln|\sin x+1| \right) + C, \quad C \in \mathbb{R}$$

(b) $$\int \frac{2\sin x+3\cos x}{\sin^2 x\cos x+9\cos^3 x}dx = \left[\begin{array}{l} R(u,v) = \frac{2u+3v}{u^2v+9v^3} \\ R(-u,-v) = R(u,v) \end{array} \middle| \; y = \text{tg}\, x \right] =$$

$$= \int \frac{2\frac{y}{\sqrt{1+y^2}}+3\frac{1}{\sqrt{1+y^2}}}{\left(\frac{y}{\sqrt{1+y^2}}\right)^2 \cdot \frac{1}{\sqrt{1+y^2}}+9\left(\frac{1}{\sqrt{1+y^2}}\right)^3} \cdot \frac{dy}{1+y^2} = \int \frac{\frac{2y+3}{\sqrt{1+y^2}}}{\frac{y^2+9}{(\sqrt{1+y^2})^3}} \cdot \frac{dy}{1+y^2} = \int \frac{2y+3}{y^2+9}dy = \left[\begin{array}{l} \Delta < 0 \\ (y^2+9)' = 2y \end{array} \right] =$$

$$= \int \frac{2ydy}{y^2+9} + \int \frac{3dy}{y^2+9} = \ln(y^2+9) + \text{arctg}\left(\frac{y}{3}\right) = \ln(\text{tg}^2 x+9) + \text{arctg}\left(\frac{\text{tg}\, x}{3}\right) + C, \quad C \in \mathbb{R}$$

Obliczenia pomocnicze:

$$\int \frac{2y}{y^2+9}dy = \left[\begin{array}{l} z = y^2+9 \\ dz = 2ydy \end{array} \right] = \int \frac{dz}{z} = \ln|z| = \ln(y^2+9)$$

$$\int \frac{3dy}{y^2+9} = \int \frac{3dy}{9\left(\left(\frac{y}{3}\right)^2+1\right)} = \left[\begin{array}{l} z = \frac{y}{3} \\ dz = \frac{1}{3}dy \end{array} \right] = \int \frac{dz}{z^2+1} = \text{arctg}z = \text{arctg}\left(\frac{y}{3}\right)$$

(c) $$\int \frac{1}{\sin x + \cos x} dx = \left[\begin{array}{l|l} R(u,v) = \dfrac{1}{u+v} & \\ \text{nie spełnia żadnego} & y = \operatorname{tg}\left(\dfrac{x}{2}\right) \\ \text{warunku szczególnego} & \end{array} \right] = \int \frac{\frac{2dy}{1+y^2}}{\frac{2y}{1+y^2} + \frac{1-y^2}{1+y^2}} = -2 \int \frac{dy}{y^2-2y-1} =$$

$$= \left[\begin{array}{l} \Delta = 8,\ t_{1,2} = 1 \pm \sqrt{2},\ y^2-2y-1 = (y-1-\sqrt{2})(y-1+\sqrt{2}) \\ \dfrac{1}{y^2-2y-1} = \dfrac{a}{y-1-\sqrt{2}} + \dfrac{b}{y-1+\sqrt{2}} \\ 1 = a(y-1+\sqrt{2}) + b(y-1-\sqrt{2}) = (a+b)y + a(-1+\sqrt{2}) + b(-1-\sqrt{2}) \\ \left\{ \begin{array}{rcrcl} a & + & b & = & 0 \\ (-1+\sqrt{2})a & + & (-1-\sqrt{2})b & = & 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcr} a & = & 1/(2\sqrt{2}) \\ b & = & -1/(2\sqrt{2}) \end{array} \right. \end{array} \right] =$$

$$= -2 \int \left(\frac{1/(2\sqrt{2})}{y-1-\sqrt{2}} + \frac{-1/(2\sqrt{2})}{y-1+\sqrt{2}} \right) dy = -\frac{1}{\sqrt{2}} \ln|y-1-\sqrt{2}| + \frac{1}{\sqrt{2}} \ln|y-1+\sqrt{2}| =$$

$$= -\frac{1}{\sqrt{2}} \ln\left|\operatorname{tg}\left(\frac{x}{2}\right) - 1 - \sqrt{2}\right| + \frac{1}{\sqrt{2}} \ln\left|\operatorname{tg}\left(\frac{x}{2}\right) - 1 + \sqrt{2}\right| + C,\ \ C \in \mathbb{R}$$