

$$\begin{aligned} \cos(z+w) &= \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} = \frac{2e^{i(z+w)} + 2e^{-i(z+w)}}{4} \\ &= \frac{e^{i(z+w)} - e^{i(z-w)} - e^{-i(w-z)} + e^{-i(z+w)}}{4} \end{aligned}$$

$$\begin{aligned} & \left( e^{i(z+w)} + e^{i(z-w)} + e^{-i(w-z)} + e^{-i(z+w)} \right) \frac{1}{4} = \\ &= \frac{e^{iz} + e^{-iz}}{2} \cdot \frac{e^{+wi} + e^{-wi}}{2} - \frac{e^{iz} - e^{-iz}}{2i} \cdot \frac{e^{iw} - e^{-iw}}{2i} = \\ &= \cos z \cdot \cos w - \sin z \cdot \sin w \end{aligned}$$

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$$\begin{aligned} \cos z \cdot \cos z + \sin z \cdot \sin z &= \frac{(e^{iz} + e^{-iz})^2}{4} + \frac{(e^{iz} - e^{-iz})^2}{-4} = \\ &= \frac{e^{2iz} + 2e^{i^2} + e^{-2iz} + e^{-2iz} - 2e^{i^2} + e^{-2iz}}{4} = 1 \end{aligned}$$