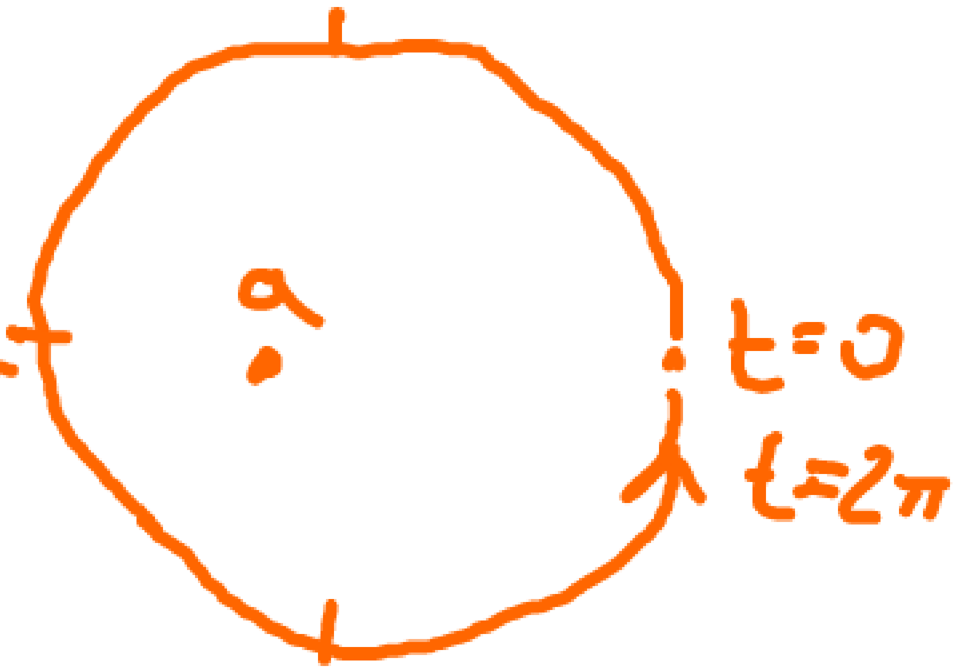
$\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$

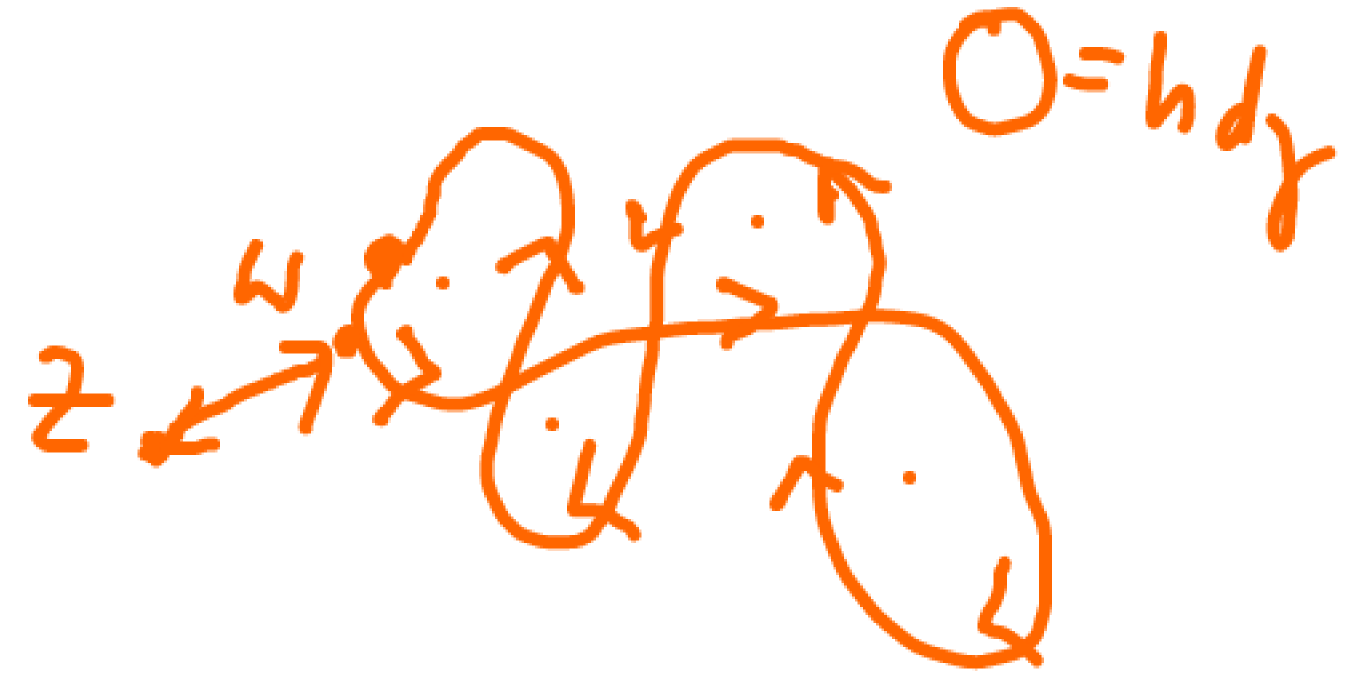
okrąg zorientowany dodatnio o śr. w $a \in \mathbb{C}$
i promieniu $r > 0$



Np.

$$\int_\gamma \frac{1}{w-a}\,dw = \int_0^{2\pi} \frac{\gamma'(t)\,dt}{\gamma(t)-a} = \int_0^{2\pi} \frac{re^{it}\, i\, dt}{\cancel{a}+re^{it}-\cancel{a}} = \int_0^{2\pi} i\,dt = 2\pi i$$

$\{ w = \gamma(t)$



Def. γ - droga zamknięta w $\mathbb{C}$

$\Omega = \mathbb{C} \setminus \gamma^*$

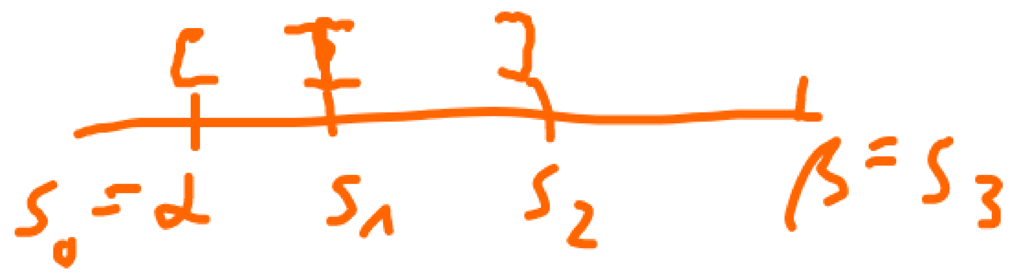
dla $z \in \Omega$ określamy <u>indeks</u> punktu z względem drogi γ wzorem

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}.$$

<u>Tw.</u> Przy zał. jw. $\mathrm{Ind}_\gamma : \Omega \to \mathbb{C}$ jest funkcją o wartościach całkowitych, stałą na każdej składowej spójnej Ω i równą zero na składowej nieograniczonej Ω.

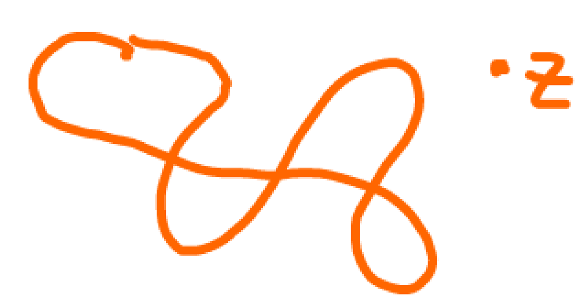
Def. $\gamma : [\alpha,\beta] \to \mathbb{C}$, $\alpha = s_0 < s_1 < \ldots < s_n = \beta$; $\gamma\restriction_{[s_{k-1}, s_k]} \in C^1([s_{k-1}, s_k])$

$S = \{s_0, s_1, \ldots, s_n\}$, ustalmy $z \in \Omega$



$$Ind_\gamma(z) = \frac{1}{2\pi i}\int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s)-z}\,ds \qquad Ind_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-z}$$

$\frac{w}{2\pi i} \in \mathbb{Z} \iff \exp(w) = 1$. Określmy



(*) $$\varphi(t) = \exp\left(\int_\alpha^t \frac{\gamma'(s)}{\gamma(s)-z}\,ds\right), \quad t \in [\alpha,\beta]$$

aby pokazać, że $Ind_\gamma(z) \in \mathbb{Z}$, wystarczy pokazać, że $\varphi(\beta) = 1$.

Różniczkujemy obustronnie (*) dla $t \in [\alpha,\beta] \setminus S$

(**) $$\varphi'(t) = \varphi(t)\frac{\gamma'(t)}{\gamma(t)-z}$$

stąd $$\left(\frac{\varphi}{\gamma - z}\right)'(t) = \frac{\varphi'(t)(\gamma(t)-z) - \varphi(t)\cdot\gamma'(t)}{(\gamma(t)-z)^2} \overset{(**)}{=} 0.$$

$\frac{\varphi}{\gamma - z} : [\alpha,\beta] \to \mathbb{C}$ cg., na każdym (s_{k-1}, s_k) ma pochodną 0

$\Rightarrow \frac{\varphi}{\gamma - z}$ stała na $[s_{k-1}, s_k]$

$\Rightarrow \frac{\varphi}{\gamma - z}$ stała na $[\alpha, \beta]$

zatem dla $t \in [\alpha,\beta]$

$$\frac{\varphi(t)}{\gamma(t)-z} = \frac{\varphi(\alpha)}{\gamma(\alpha)-z} = \frac{\exp\left(\int_\alpha^\alpha \ldots ds\right)}{\gamma(\alpha)-z} = \frac{1}{\gamma(\alpha)-z}$$

$t = \beta$: $$\varphi(\beta) = \frac{\gamma(\beta)-z}{\gamma(\alpha)-z} \underset{\gamma(\alpha)=\gamma(\beta)}{=} 1.$$

$$Ind_\gamma(z) = \frac{1}{2\pi i}\int_\alpha^\beta \frac{\gamma'(s)\,ds}{\underbrace{\gamma(s)}_{\text{cg.}} - z}$$

holom. wzgl. $z \in \mathbb{C} \setminus \gamma^*$

$\Rightarrow Ind_\gamma$ – cg. na $\mathbb{C} \setminus \gamma^*$

Ind_γ przekształca zbiory spójne w zbiory spójne

$\underbrace{Ind_\gamma(G)}_{\text{zb. spójny}} \subset \mathbb{Z}$ $\Rightarrow$ $Ind_\gamma(G)$ jest jednopunktowy

G – skł. spójna Ω

$|\gamma'(s)| \leq M$

$$|Ind_\gamma(z)| = \left|\frac{1}{2\pi i}\int_\alpha^\beta \frac{\overbrace{\gamma'(s)}^{|\cdot| \leq M}\,ds}{\underbrace{\gamma(s)-z}_{|\cdot| \geq dist(z,\gamma^*)}}\right| \leq \frac{1}{2\pi}(\beta - \alpha)\frac{M}{dist(z,\gamma^*)} \xrightarrow{|z|\to\infty} 0$$

$\to Ind_\gamma = 0$ na skł. nieogr. $\boxtimes$

<u>Np.</u> γ - dodatnio zorientowany okrąg o śr. w $a \in \mathbb{C}$ i pr. $r>0$

$$\operatorname{Ind}_\gamma(z) = \begin{cases} 0, & |z-a|>r \\ 1, & |z-a|<r \end{cases}$$

$$\operatorname{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{dw}{w-a} = \ldots (\text{było}) = \frac{1}{2\pi i}\cdot 2\pi i = 1$$

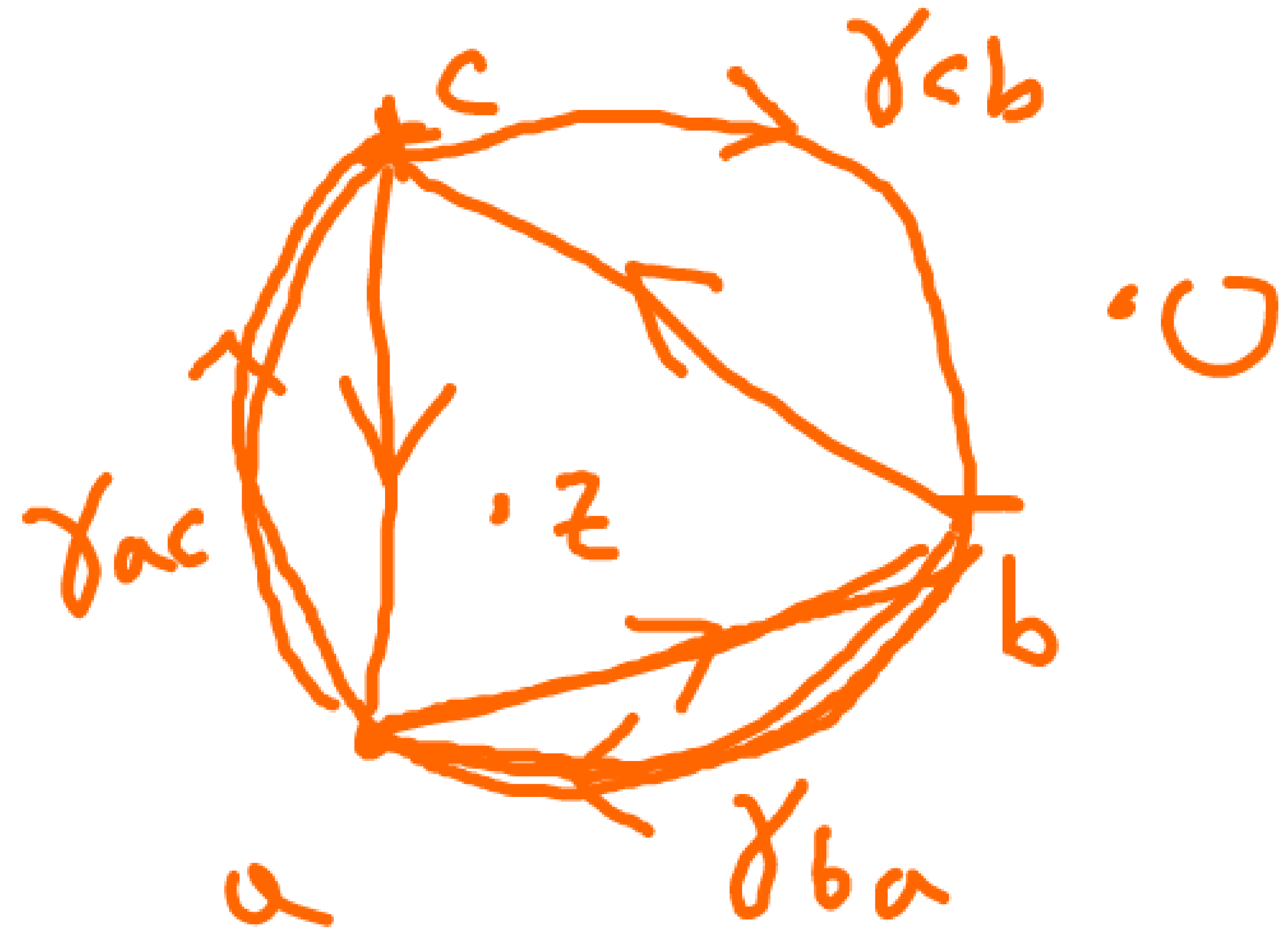
γ - ujemnie zorientowany ...

$$\operatorname{Ind}_\gamma(z) = \begin{cases} 0, & |z-a|>r \\ -1, & |z-a|<r \end{cases}$$

Np. $a,b,c \in \mathbb{C}$ niewspółliniowe

zob. $\partial\Delta(a,b,c)$ jest dod. zorientowany

$$\partial\Delta(a,b,c) = \langle a,b\rangle + \langle b,c\rangle + \langle c,a\rangle$$



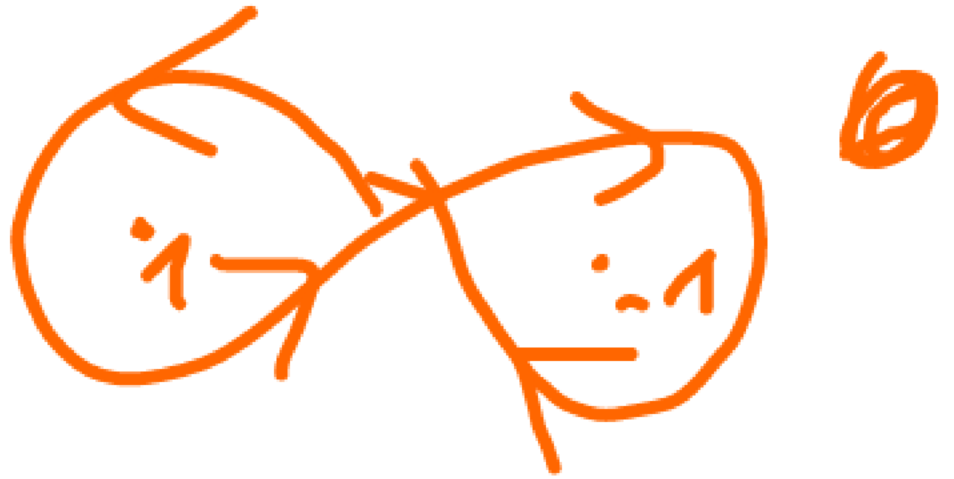
$\gamma = \partial\Delta(a,b,c)$

$\mathrm{Ind}_\gamma(z) = ?$

$C := \gamma_{cb} + \gamma_{ba} + \gamma_{ac}$ = ujemnie zor. okrąg opisany na $\Delta(a,b,c)$

$$\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \ldots = \frac{1}{2\pi i}\Big(\underbrace{\int_{\langle a,b\rangle} + \int_{\gamma_{ba}}}_{\gamma_1} + \underbrace{\int_{\langle b,c\rangle} + \int_{\gamma_{cb}}}_{\gamma_3} + \underbrace{\int_{\langle c,a\rangle} + \int_{\gamma_{ac}}}_{\gamma_2} - \int_C\Big)\ldots$$

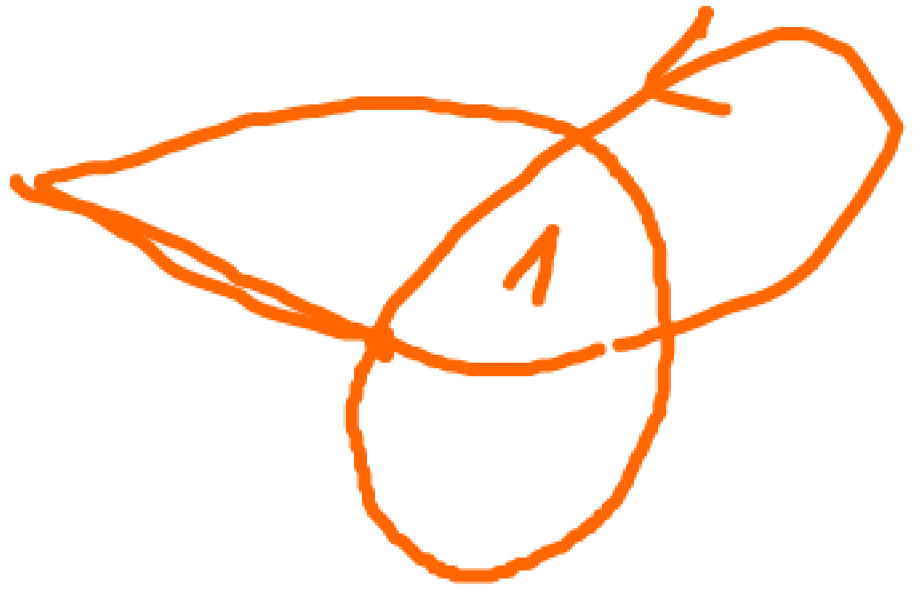
$$= \underbrace{\mathrm{Ind}_{\gamma_1}(z)}_{0} + \underbrace{\mathrm{Ind}_{\gamma_3}(z)}_{0} + \underbrace{\mathrm{Ind}_{\gamma_2}(z)}_{0} - \underbrace{\mathrm{Ind}_{C}(z)}_{=-1} = 1$$



. . .

Ind = 0

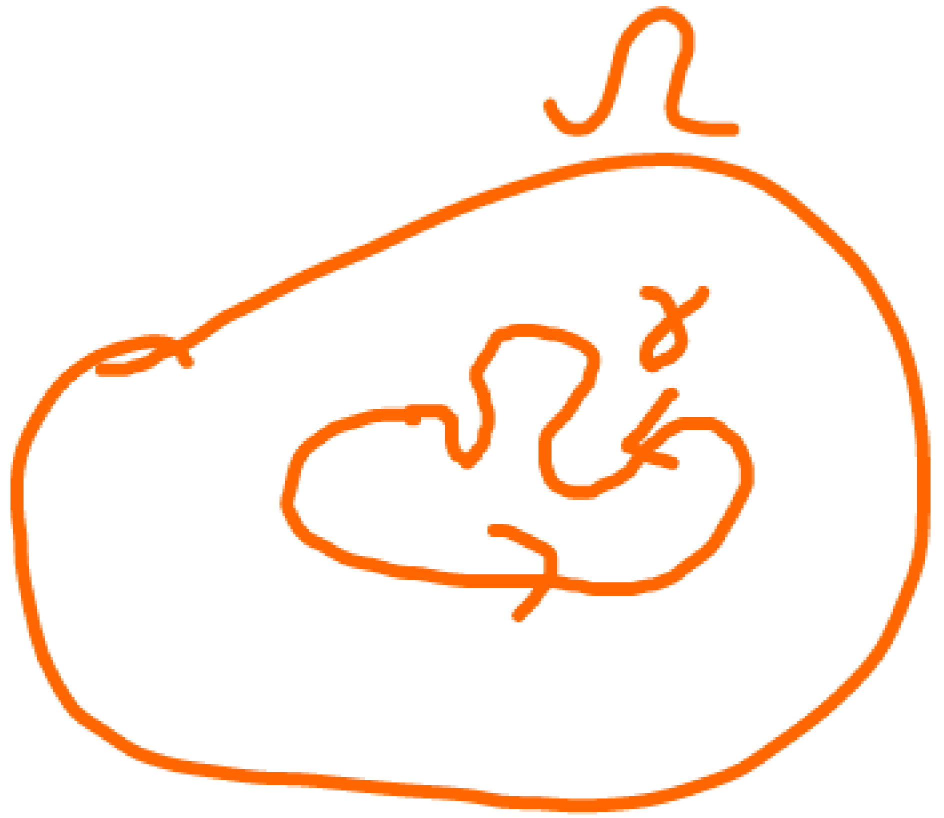
$\gamma(t) = a + re^{it}$, $t \in [0, 4\pi]$



Tw. $F \in H(\Omega)$, $F' \in C(\Omega)$ $\Rightarrow$ $\int_\gamma F'(z)\,dz = 0$

dla dow. drogi zamkniętej γ, $\gamma^* \subset \Omega$

(ćw.)



$\gamma^* \subset \Omega$

Wniosek

γ - droga zamknięta w $\mathbb{C}$

$$\int_\gamma z^n\,dz = 0 \quad \text{dla } n = 0, 1, 2, \ldots, \text{ bo } z^n = \Big(\underbrace{\frac{z^{n+1}}{n+1}}_{\in H(\mathbb{C})}\Big)'$$

Tw. (Cauchy'ego dla Δ)

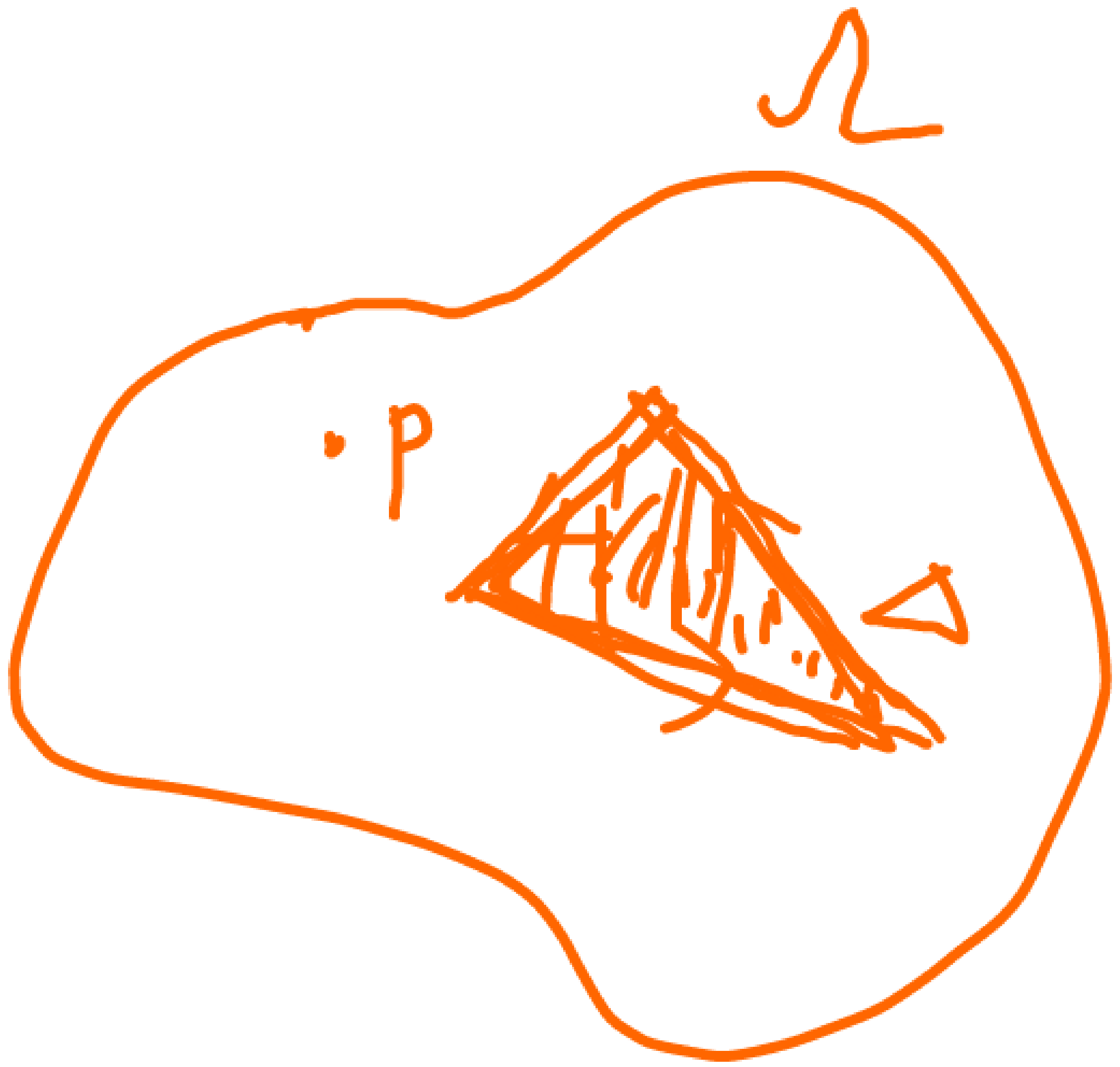
$\Omega \subset \mathbb{C}$ zb. otwarty, Δ - trójkąt (zb. wypukły, domknięty) zawarty w Ω

$p \in \Omega$

Zał. $f \in \underline{C(\Omega)}$, $f \in \underline{H(\Omega \setminus \{p\})}$

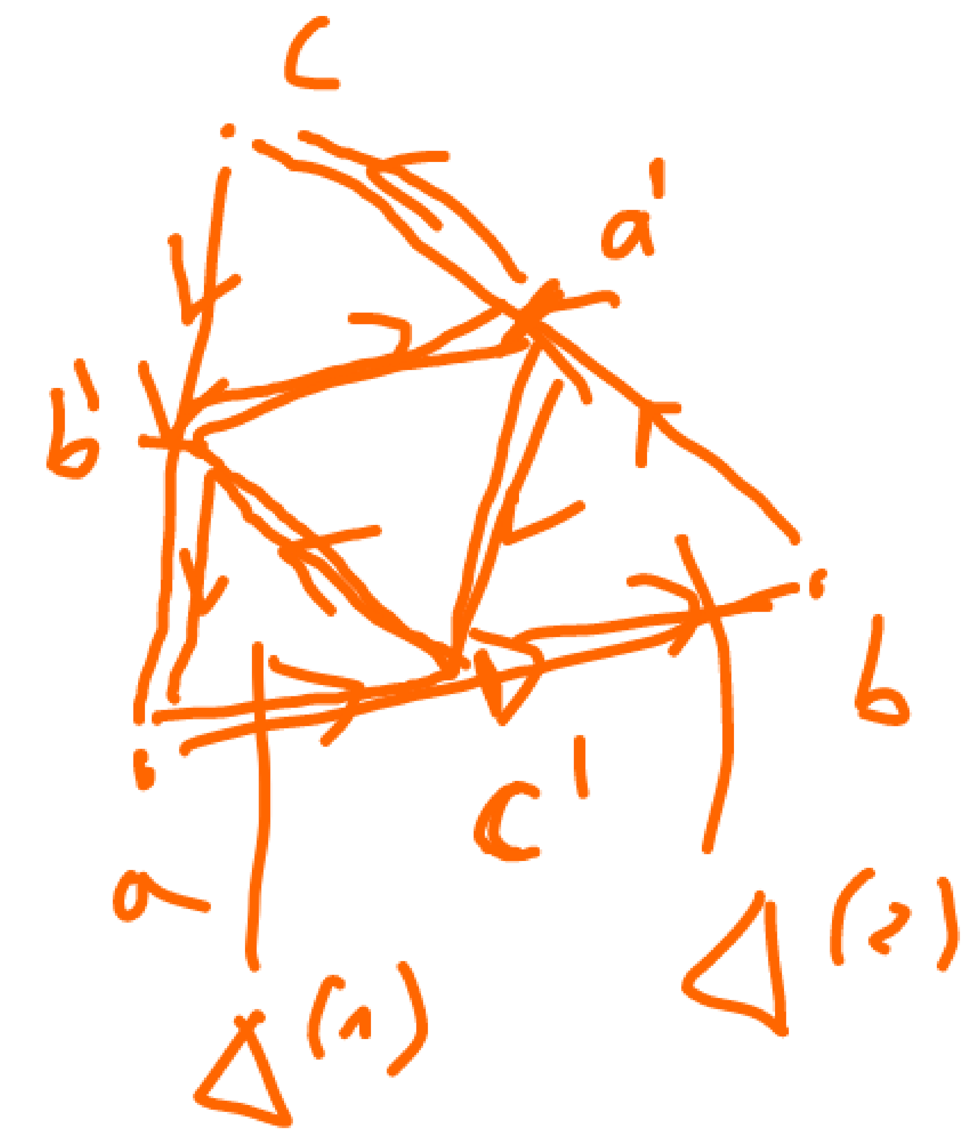
wtedy

$$\int_{\partial\Delta} f(z)\,dz = 0.$$



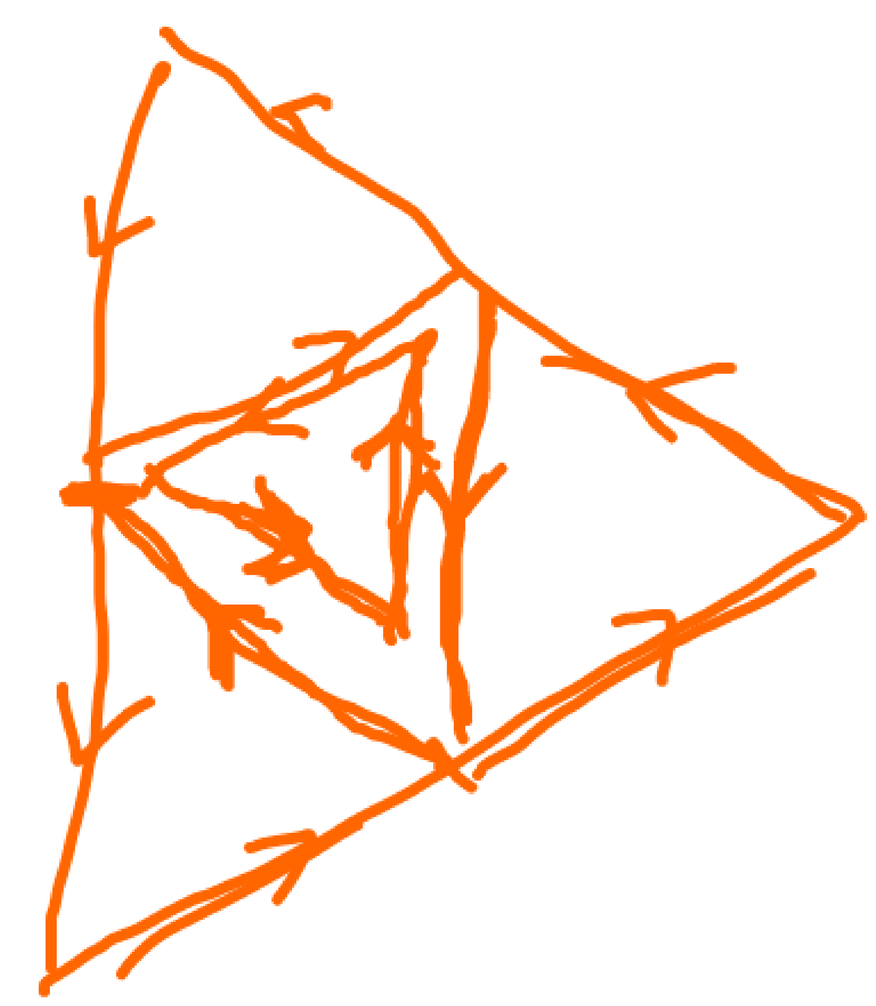
Dowód. Załóżmy najpierw, że $p \notin \Delta$. $\Delta = \Delta(a,b,c)$

$$\Delta^{(1)} = \Delta(a, c', b')$$
$$\Delta^{(2)} = \Delta(b, a', c')$$
$$\Delta^{(3)} = \Delta(c, b', a')$$
$$\Delta^{(4)} = \Delta(a', b', c')$$



Zauważmy, że $\displaystyle\int_{\partial\Delta} \dots = \sum_{j=1}^{4} \int_{\partial\Delta^{(j)}} \dots$

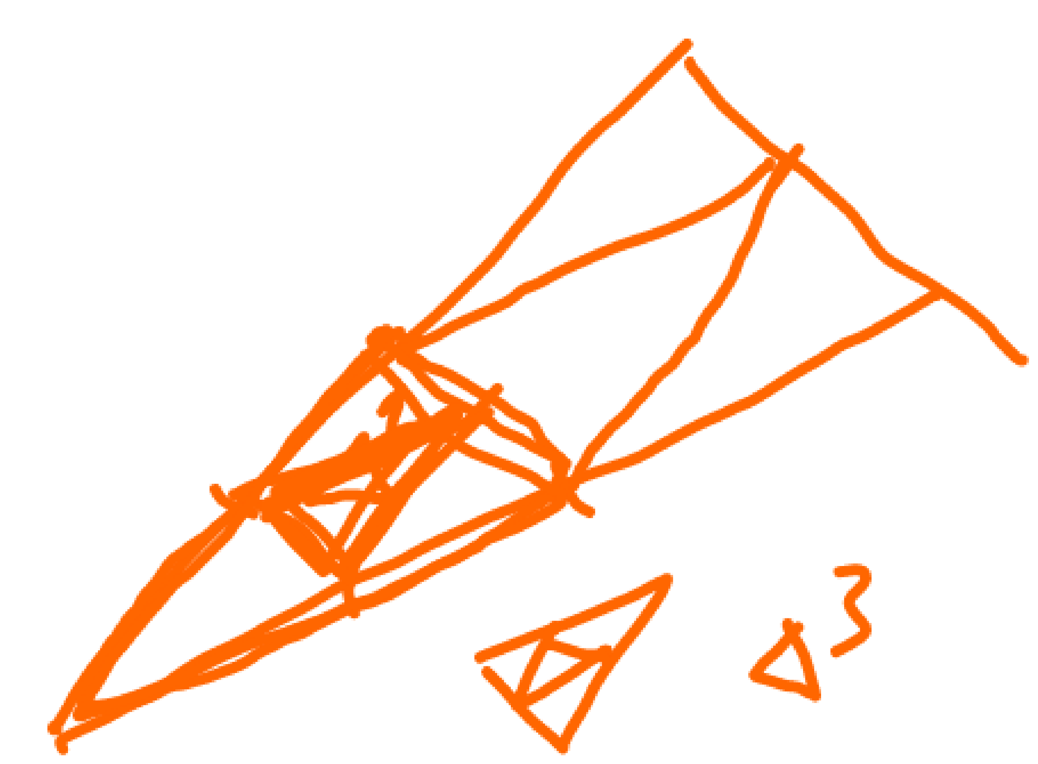
$$|J| := \left|\int_{\partial\Delta} f(z)\,dz\right| \leq \underbrace{\sum_{j=1}^{4} \left|\int_{\partial\Delta^{(j)}} f(z)\,dz\right|}_{\text{suma 4 składników}} \longrightarrow$$



Oznaczmy przez Δ^1 który z $\Delta^{(1)}, \dots, \Delta^{(4)}$, dla którego $\displaystyle\left|\int_{\partial\Delta^{(j)}} f(z)\,dz\right| \geq \frac{|J|}{4}$.

Podobnie określamy $\Delta^2, \Delta^3, \dots$

$$\left|\int_{\Delta^k} f(z)\,dz\right| \geq |J|/4^k$$



Δ^3

Otrzymujemy zstępujący ciąg $\Delta \supset \Delta_1 \supset \Delta_2 \supset \Delta_3 \supset \ldots$ (z_1, z_2, z_3, ... $\to$) podc. zbieżny do z_0

$$\ell(\partial \Delta_k) = \frac{1}{2^k}\,\ell(\Delta) =: \frac{1}{2^k} L.$$

obwód Δ_k

Istnieje dokł. jeden $z_0 \in \bigcap_{k=1}^{\infty} \Delta_k \subset \Delta$ i $p \notin \Delta$

więc istnieje $f'(z_0)$. Weźmy $\varepsilon > 0$. Istnieje $r > 0$ t. że dla $|z - z_0| < r$:

$$(*) \quad |f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0|$$

$\to$ $|z - z_0| \leq \operatorname{diam} \Delta_n \leq \ell(\Delta_n) = \frac{1}{2^n} L \quad z \in \Delta_n$

dla n dużych: $|z - z_0| < r$, gdy $z \in \Delta_n$

Δ_m

$$\left| \int_{\partial \Delta_n} f(z)\,dz \right| = \left| \int_{\partial \Delta_n} \big(f(z) - f(z_0) - f'(z_0)(z - z_0)\big)\,dz \right|$$

$$\left|\int_{\partial\Delta_n} f(z)\,dz\right| \leq \underline{\left|\int_{\partial\Delta_n} \big(f(z)-f(z_0)-f'(z_0)(z-z_0)\big)\,dz\right|} \leq$$

$$\left\{ \begin{array}{l} |g| \leq M \Rightarrow \\ \left|\int_\gamma g(z)\,dz\right| \leq M\cdot \ell(\gamma) \end{array}\right. \qquad \ell(\gamma) := \int_\alpha^\beta |\gamma'(t)|\,dt, \quad \gamma : [\alpha,\beta] \to \mathbb{C}$$

$$\leq \ell(\partial\Delta_n)\cdot \sup_{\underline{z\in\Delta_n}} |f(z)-f(z_0)-f'(z_0)(z-z_0)| \leq$$

$$\leq \underbrace{\ell(\partial\Delta_n)}_{=\frac{1}{2^n}\cdot L}\cdot \varepsilon\,|z-z_0| \leq \frac{1}{2^n}\cdot L\cdot\varepsilon\cdot\frac{1}{2^n}L = $$

$$= \underline{\varepsilon\,\frac{1}{4^n}L^2}$$

Ale $\left|\int_{\Delta_n} f(z)\,dz\right| \geq \underline{\frac{|J|}{4^n}} \Rightarrow \frac{|J|}{4^n} \leq \varepsilon\,\frac{1}{4^n}L^2$

$|J| \leq \varepsilon L^2$ z dow. ε : $|J| = 0$, $J = 0$

$$J = \int_{\partial\Delta} f(z)\,dz \ldots$$