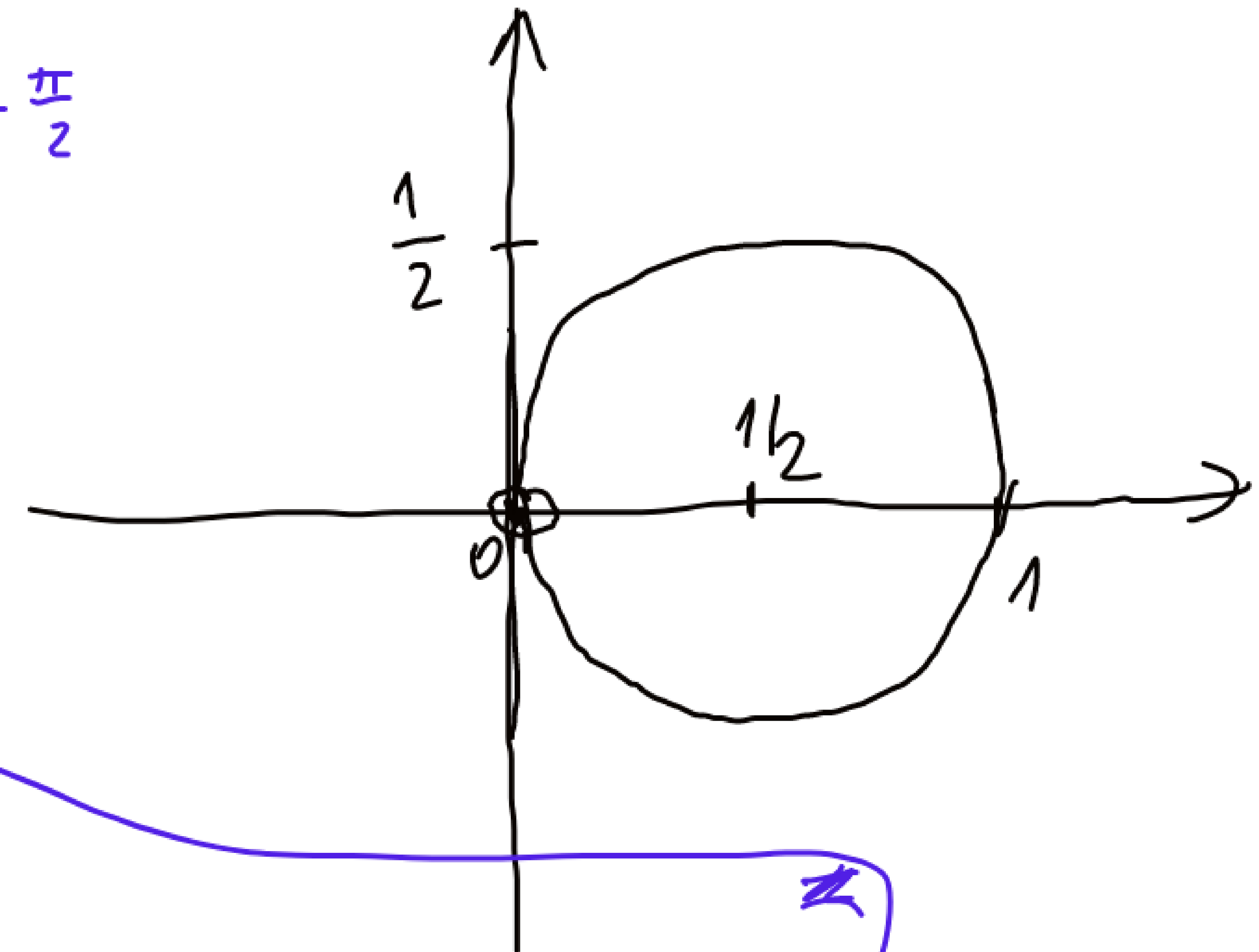
$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = (*)$$

$$D = \{(x,y):\ x^2+y^2 \le x\}$$

$$x^2 - x + y^2 \le 0$$

$$\left(x-\tfrac{1}{2}\right)^2 - \tfrac{1}{4} + y^2 \le 0$$

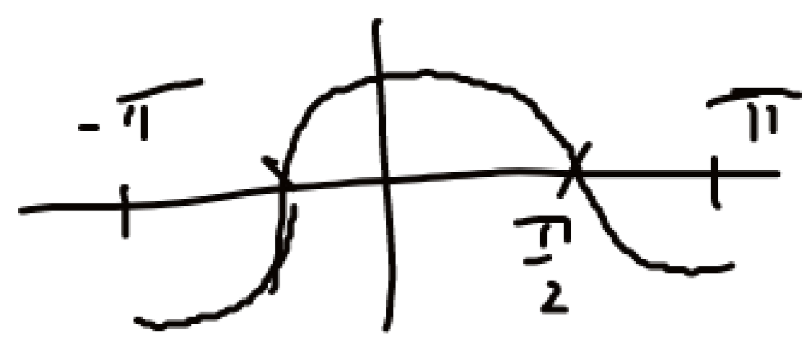
$$\left(x-\tfrac{1}{2}\right)^2 + y^2 \le \left(\tfrac{1}{2}\right)^2$$



$$\begin{cases} x = r\cos\varphi & r \geq 0 \\ y = r\sin\varphi & \varphi \in [-\pi, \pi) \end{cases}$$

$$r^2\cos^2\varphi + r^2\sin^2\varphi \le r\cos\varphi$$

$$r^2 \le r\cos\varphi \quad \Big|\, : r>0$$



$$\begin{cases} 0 \le r \le \cos\varphi \\ \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad (\text{bo } \cos\varphi \geq 0) \end{cases}$$

$$(*) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} \sqrt{r^2}\; r\, dr =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \cdot \frac{r^3}{3}\Big|_{r=0}^{r=\cos\varphi} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3\varphi}{3}\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2\varphi\ \cos\varphi\ d\varphi}{3} = \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \\ t \in [-1, 1] \end{array} \right|$$

$$= \int_{-1}^{1} (1-t^2)\,\frac{dt}{3} = \frac{1}{3}\left(t - \frac{t^3}{3}\right)\Big|_{t=-1}^{t=1} = \frac{1}{3}\left(1 - \frac{1}{3} - \left(-1 - \frac{(-1)^3}{3}\right)\right) = \frac{4}{9}$$

$$= \frac{1}{3}\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\Big|_{\varphi=-\frac{\pi}{2}}^{\varphi=\frac{\pi}{2}}$$

$$\cancel{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dots d\varphi \overset{?}{=} \int \dots dt}$$

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$$\int \frac{\cos^2\varphi\,\cos\varphi}{3}\, d\varphi = \left| t = \sin\varphi \right| = \int (1-t^2)\,\frac{dt}{3} =$$

$$= \frac{1}{3}\left(\sin\varphi - \frac{\sin^3\varphi}{3}\right) + C$$