$$\iint_D \frac{x \operatorname{arctg}\frac{y}{x}}{x^2+y^2}\,dx\,dy = (*)$$

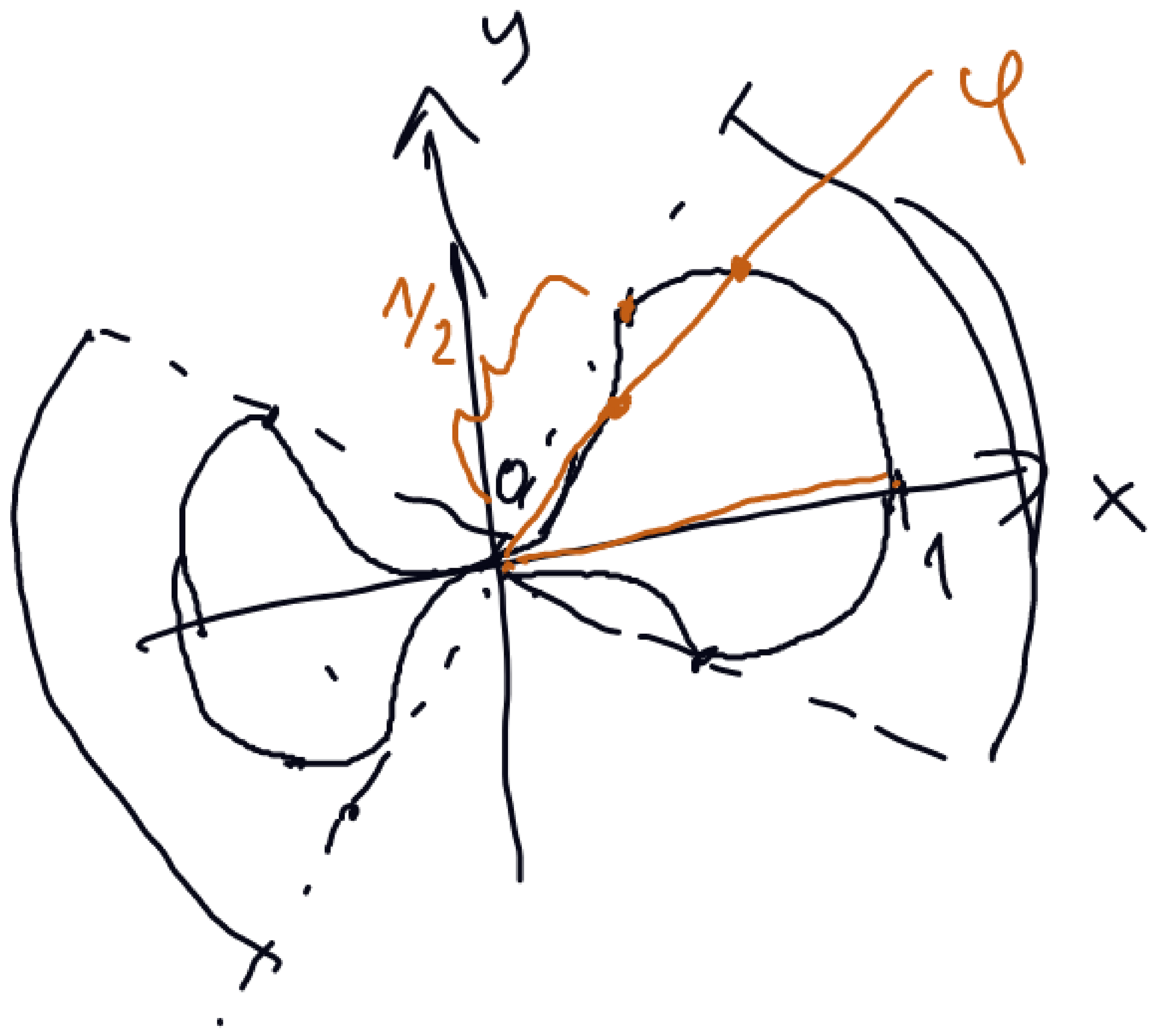
$$D = \{ y^2 \le (\underbrace{x^2+y^2}_{r^2})^{3/2} \le x^2 \}$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad (r \ge 0) \quad \varphi \in \left[-\tfrac{\pi}{2}, \tfrac{3}{2}\pi\right]$$

$$r^2\sin^2\varphi \le r^3 \ \wedge\ r^3 \le r^2\cos^2\varphi$$

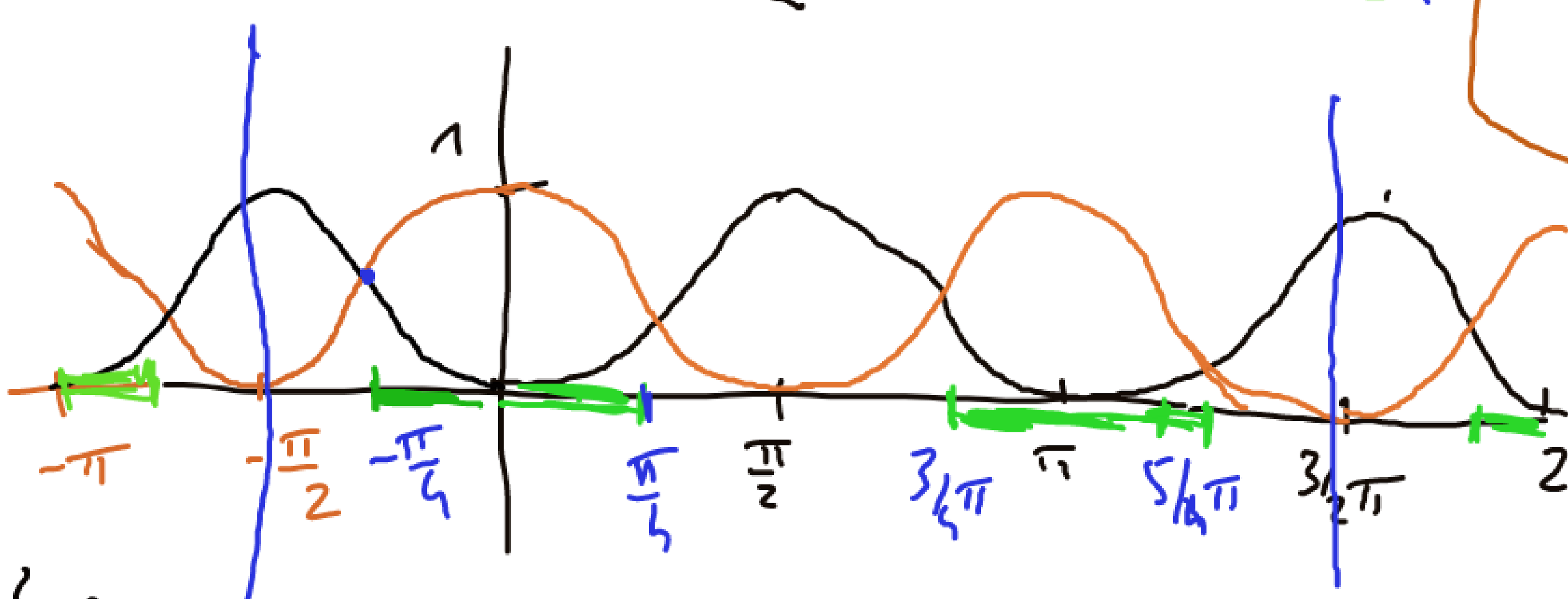
Dzielimy obustronnie przez $r > 0$:

$$\sin^2\varphi \le r \ \wedge\ r \le \cos^2\varphi$$



$$\boxed{\sin^2\varphi \le r \le \cos^2\varphi}$$

$$\sin^2\varphi \le \cos^2\varphi \quad \boxed{\varphi \in \left[-\tfrac{\pi}{4}, \tfrac{\pi}{4}\right] \cup \left[\tfrac{3}{4}\pi, \tfrac{5}{4}\pi\right]}$$



$$(*) = \left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi}\right) d\varphi \int_{\sin^2\varphi}^{\cos^2\varphi} \frac{r\cos\varphi \operatorname{arctg}\frac{r\sin\varphi}{r\cos\varphi}}{\underbrace{r^2\cos^2\varphi + r^2\sin^2\varphi}_{= r^2}}\, r\, dr =$$

$$= \left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi}\right) d\varphi\ \cos\varphi \operatorname{arctg}(\operatorname{tg}\varphi) \int_{\sin^2\varphi}^{\cos^2\varphi} 1\, dr = \left(\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} + \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi}\right) \cos\varphi \operatorname{arctg}(\operatorname{tg}\varphi)(\cos^2\varphi - \sin^2\varphi)\, d\varphi$$

$$\operatorname{arctg}(\operatorname{tg}\varphi) = ?$$

$$\left\{ \operatorname{arctg}(y) = x \iff x \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \ ; \ \operatorname{tg} x = y = \operatorname{tg}\varphi \right.$$

Jeśli $\varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, to $x = \varphi$, czyli $\operatorname{arctg}(\operatorname{tg}\varphi) = \varphi$

Jeśli $\varphi \in \left[\frac{3}{4}\pi, \frac{5}{4}\pi\right]$: $\operatorname{tg} \underset{\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}{\boxed{x}} = \operatorname{tg}\varphi = \operatorname{tg}(\underbrace{\boxed{\varphi - \pi}}_{\in\left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}) \Rightarrow x = \varphi - \pi$

$$\operatorname{arctg}(\operatorname{tg}\varphi) = \varphi - \pi$$

$$(*) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos\varphi \cdot \varphi \cdot (\cos^2\varphi - \sin^2\varphi)\, d\varphi + \int_{3/4\pi}^{\frac{5}{4}\pi} \cos\varphi \cdot (\varphi - \pi) \cdot (\cos^2\varphi - \sin^2\varphi)\, d\varphi$$

$$\int \cos\varphi(\cos^2\varphi - \sin^2\varphi)\, d\varphi$$