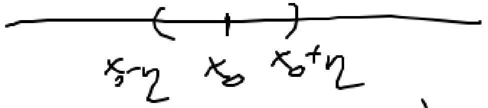
Pochodne

Załóżmy, że $f: D \to \mathbb{R}$, $x_0 \in D$ oraz dla pewnej $\eta > 0$ zachodzi $(x_0-\eta, x_0+\eta) \subset D$.



Załóżmy, że istnieje granica

$$g = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}.$$

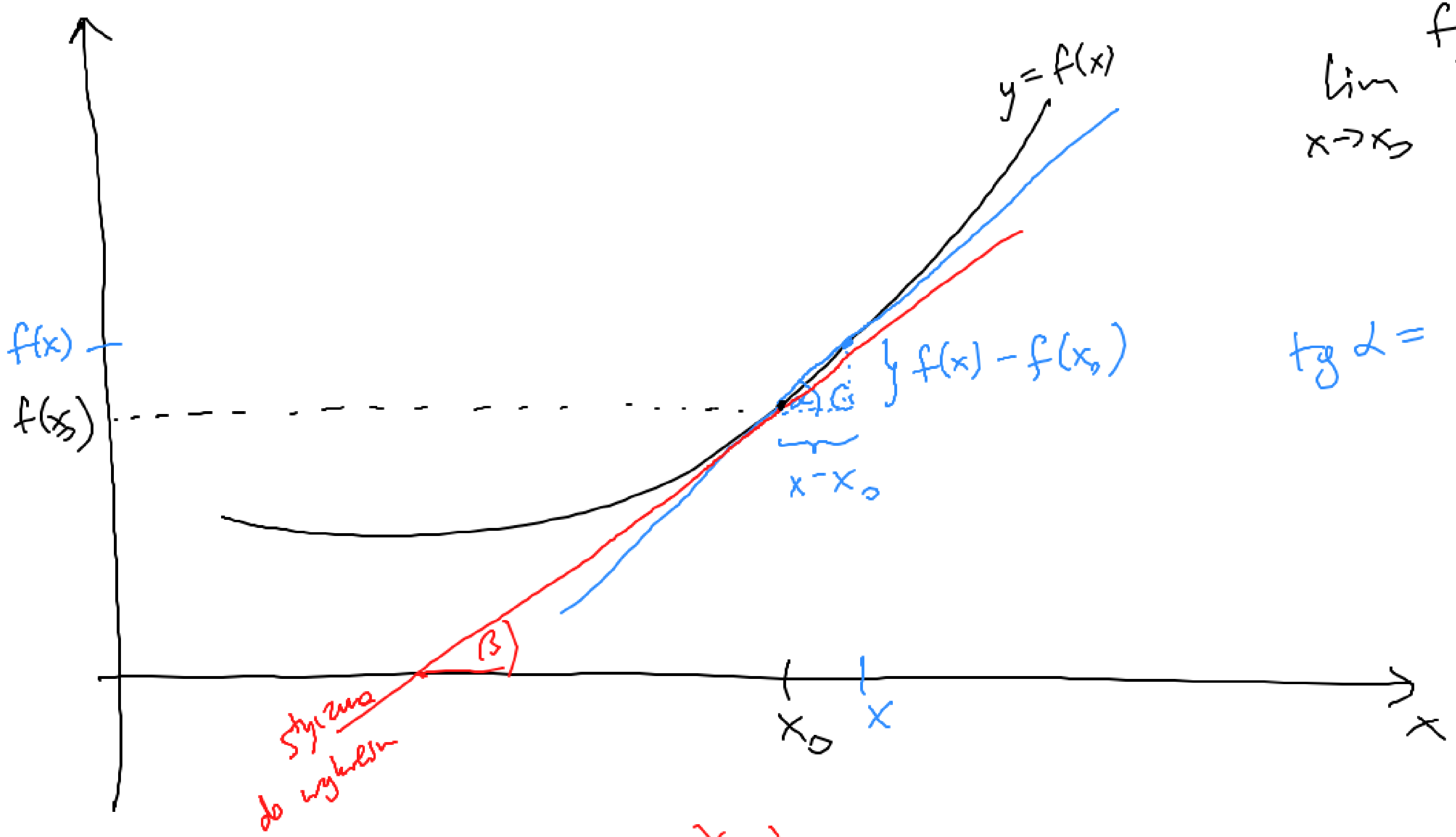
Wówczas mówimy, że f ma pochodną (właściwą, gdy $g \in \mathbb{R}$, niewłaściwą, gdy $g \in \{-\infty, \infty\}$) w punkcie x_0 i oznaczamy

$$f'(x_0) = g.$$

Inne oznaczenia: $f'(x_0) = \frac{df}{dx}(x_0) = f_x(x_0) = Df(x_0)$.

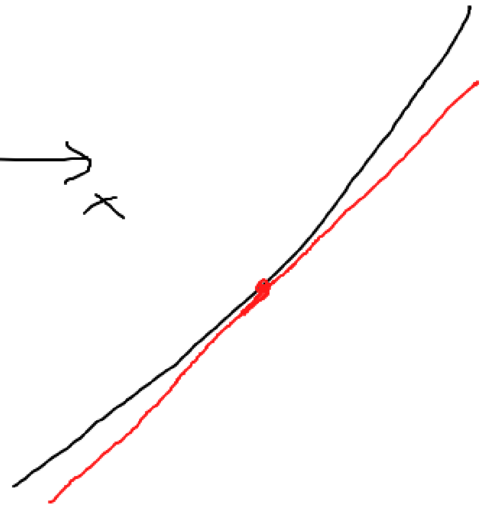
Mówimy, że f jest różniczkowalna w x_0, jeśli ma w x_0 pochodną właściwą.

$$\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$



$$\operatorname{tg}\alpha = \frac{f(x)-f(x_0)}{x-x_0}$$

$$\operatorname{tg}\beta = f'(x_0)$$



$f(t)$ = położenie cząstki w chwili t (na prostej)

→ różnica położenia cząstki od chwili t_0 do t (droga, którą przebyła cząstka, przy założeniu, że nie zawracała)

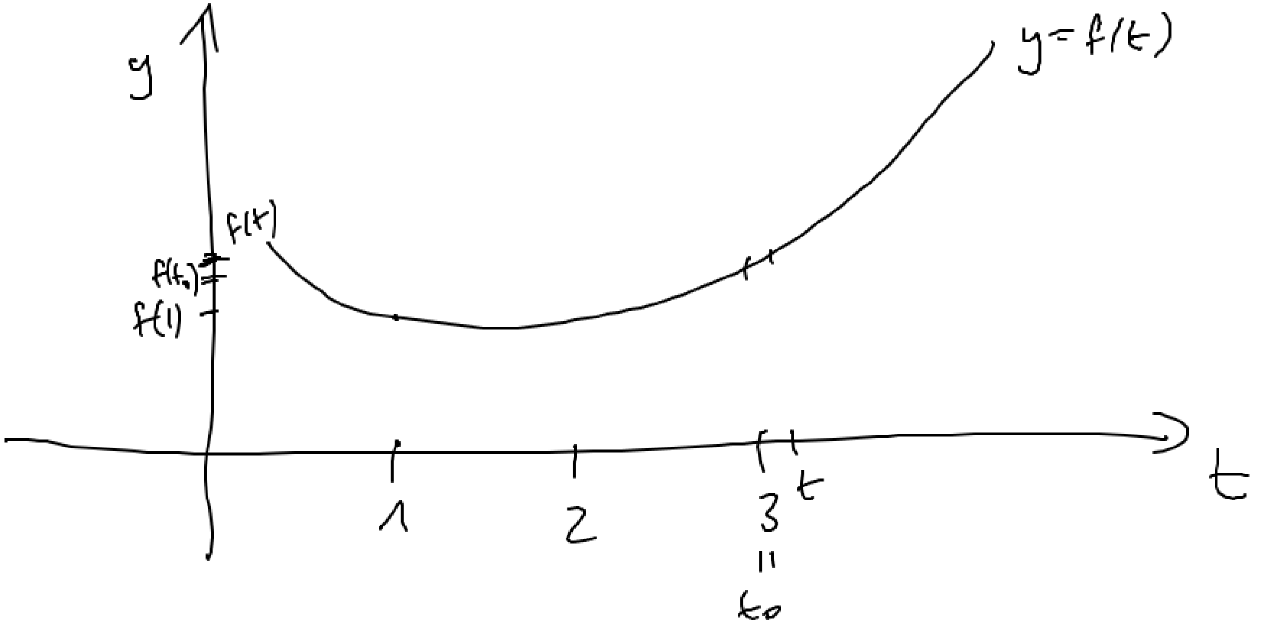
$\dfrac{f(t)-f(t_0)}{t-t_0}$ = prędkość średnia od ~~t_0~~ do t w czasie

czas, który upłynął od chwili t_0 do t

$$\lim_{t\to t_0}\frac{f(t)-f(t_0)}{t-t_0} = \text{prędkość chwilowa cząstki w } t_0$$

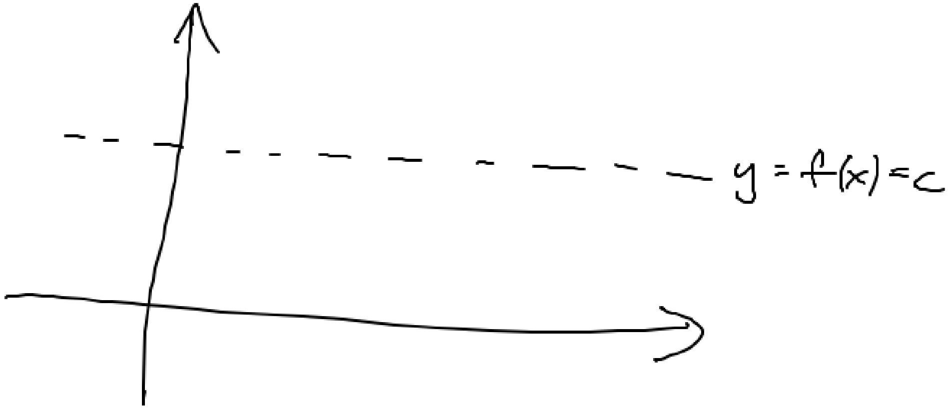
$$\|$$

$$f'(t_0)$$



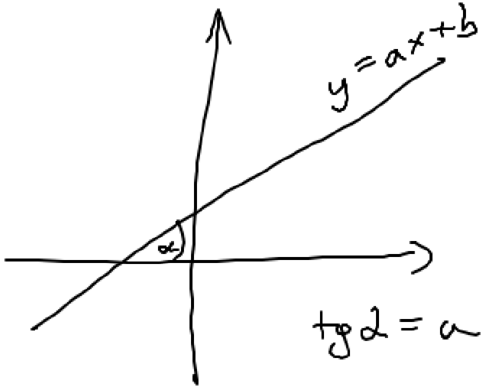
Przykłady

- $f(x) = c$ (f. stała), $x \in \mathbb{R}$



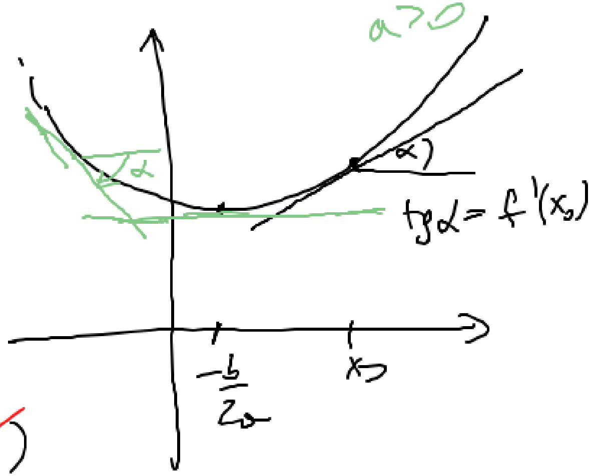
$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \underbrace{\boxed{\frac{c - c}{x - x_0}}}_{=0} = \lim_{x \to x_0} 0 = 0 .$$

- $f(x) = ax + b$ ($a, b \in \mathbb{R}$ - stałe), $x \in \mathbb{R}$



$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{ax + b - (ax_0 + b)}{x - x_0} = \lim_{x \to x_0} \frac{a(x - x_0)}{x - x_0} = a$$

$\cdot \quad f(x) = ax^2 + bx + c \quad , x \in \mathbb{R} \qquad (a, b, c \in \mathbb{R} \text{ - stałe})$



$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{(ax^2+bx+c) - (ax_0^2 + bx_0 + c)}{x - x_0} =$$

$$= \lim_{x \to x_0} \frac{a(x^2 - x_0^2) + b(x - x_0)}{x - x_0} = \lim_{x \to x_0} \frac{a\cancel{(x - x_0)}(x + x_0) + b\cancel{(x - x_0)}}{\cancel{x - x_0}} =$$

$$= \lim_{x \to x_0} \left[a(\underbrace{x}_{x_0} + x_0) + b \right] = 2ax_0 + b .$$

$$f'(x_0) = 0 \iff 2ax_0 + b = 0$$

$$x_0 = -\frac{b}{2a} \qquad (\text{zał. } a \neq 0)$$

Wzory:

- $(c)' = 0$ (f. stała)
- $(x^n)' = n x^{n-1}$ $(n \in \mathbb{R},\ x > 0)$
- $(a^x)' = a^x \ln a$ $(a > 0,\ x \in \mathbb{R})$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$

$\Big\}\ x \in \mathbb{R}$

- $(\ln x)' = \frac{1}{x}$ $(x > 0)$

w szczególności:

$(x^0)' = (1)' = 0$

$(x^1)' = (x)' = 1$

$(x^2)' = 2x^{2-1} = 2x$

Np. sprawdzimy $\sin'(x_0)$: dla $x_0 = 0$:

$$\sin'(x_0) = \lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0} = \lim_{x \to 0} \frac{\sin x}{x} = 1$$

$\sin(a+b) = \sin a \cos b + \sin b \cos a$

Dla dowolnego x_0:

$$\sin'(x_0) = \lim_{x\to x_0} \frac{\sin x - \sin x_0}{x - x_0} = \left| \begin{array}{l} x = x_0 + h \\ h = x - x_0 \end{array} \right| = \lim_{h\to 0} \frac{\sin(x_0+h) - \sin(x_0)}{h} =$$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos 2a = \cos^2 a - \sin^2 a$

$\cos h = \cos^2 \frac{h}{2} - \sin^2 \frac{h}{2}$

$$= \lim_{h\to 0} \frac{\sin x_0 \cos h + \sin h \cos x_0 - \sin x_0}{h} =$$

$$= \lim_{h\to 0} \left[\sin x_0 \underbrace{\frac{(\cos h - 1)}{h}}_{\to 0} + \underbrace{\frac{\sin h}{h}}_{\to 1} \cos x_0 \right] = \underline{\cos x_0}$$

Policzmy osobno

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{\left(\cos^2\frac{h}{2} - \sin^2\frac{h}{2}\right) - \left(\cos^2\frac{h}{2} + \sin^2\frac{h}{2}\right)}{h} = \lim_{h\to 0} -2\underbrace{\boxed{\frac{\sin^2\frac{h}{2}}{\left(\frac{h}{2}\right)^2}}}_{\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)^2 \to 1} \cdot \frac{1}{4}\underbrace{h}_{\to 0} = 0$$

Tw. Jeśli $f, g : D \to \mathbb{R}$ mają pochodne właściwe w punkcie $x_0 \in D$, to

$$(f+g)'(x_0) = f'(x_0) + g'(x_0),$$

$$(f-g)'(x_0) = f'(x_0) - g'(x_0),$$

$$(f \cdot g)'(x_0) = f'(x_0)\, g(x_0) + f(x_0) \cdot g'(x_0)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) g(x_0) - f(x_0) g'(x_0)}{g^2(x_0)}, \quad \text{o ile } g(x_0) \neq 0$$

Dow.

Np.

$\boxed{(fg)'(x_0) = f'(x_0)\,g(x_0) + f(x_0)\,g'(x_0)}$ $\quad (fg)' = f'g + fg'$

$$(2x^2+3x)' = \underline{(2x^2)'} + (3x)' = \underline{(2)'\cdot x^2 + 2\cdot(x^2)'} + 3'\cdot x + 3\cdot(x)' =$$

$$= 0\cdot x^2 + 2\cdot 2x + 0\cdot x + 3\cdot 1 =$$

$$= \underline{4x+3}$$

$$(x^n)' = n\,x^{n-1}$$

$$(x^2)' = 2x^{2-1} = 2x^1 = 2x$$

$$(x)' = (x^1)' = 1\cdot x^{1-1} = 1\cdot x^0 = 1$$

Dokładny wzór:

$$(c\cdot f)'(x_0) = c\cdot f'(x_0) \qquad (c - \text{stała})$$

Uzasadnienie:

$$(c\cdot f)'(x_0) = \underbrace{c'}_{=0}\cdot f(x_0) + c\cdot f'(x_0) = c\,f'(x_0)$$

№1. $(x>0)$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\left(\frac{\sin x + x}{\sqrt{x}}\right)' = \frac{(\sin x + x)' \cdot \sqrt{x} - (\sin x + x)\cdot(\sqrt{x})'}{(\sqrt{x})^2} =$$

$$= \frac{((\sin x)' + (x)') \cdot \sqrt{x} - (\sin x + x)\left(x^{\frac{1}{2}}\right)'}{x} =$$

$$(x^n)' = nx^{n-1}$$

$$\sqrt{x}' = \left(x^{\frac{1}{2}}\right)' = \frac{1}{2} x^{\frac{1}{2}-1} =$$

$$= \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2}\,\frac{1}{x^{\frac{1}{2}}} =$$

$$= \frac{1}{2\sqrt{x}}$$

$$= \frac{(\cos x + 1)\cdot\sqrt{x} - (\sin x + x)\cdot\frac{1}{2\sqrt{x}}}{x}.$$

$$\left(x^2e^x - \sqrt[3]{x}\cdot\cos x\right)' = \left(x^2e^x\right)' - \left(\sqrt[3]{x}\cdot\cos x\right)' =$$

$$= \left(x^2\right)'e^x + x^2\left(e^x\right)' - \left[\left(\sqrt[3]{x}\right)'\cdot\cos x + \sqrt[3]{x}\cdot(\cos x)'\right] =$$

$$= 2xe^x + x^2e^x - \left[\frac{1}{3}x^{-\frac{2}{3}}\cdot\cos x + \sqrt[3]{x}\cdot(-\sin x)\right]$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\left(a^x\right)' = a^x \ln a$$

$$\left(e^x\right)' = e^x \ln e = e^x \cdot 1 =$$

$$= e^x$$

$$\boxed{\left(e^x\right)' = e^x}$$

$$\left(x^n\right)' = nx^{n-1}$$

$$\left(x^{\frac{1}{3}}\right)' = \frac{1}{3}x^{\frac{1}{3}-1} =$$

$$= \frac{1}{3}x^{-2/3}$$

$$(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cdot\cos x - \sin x\cdot(\cos x)'}{\cos^2 x} =$$

$$= \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= 1 + \operatorname{tg}^2 x$$

$$\bullet \quad (\operatorname{ctg} x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \cdot \sin x - \cos x\,(\sin x)'}{\sin^2 x} =$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$(\sin x)' = \cos x$$

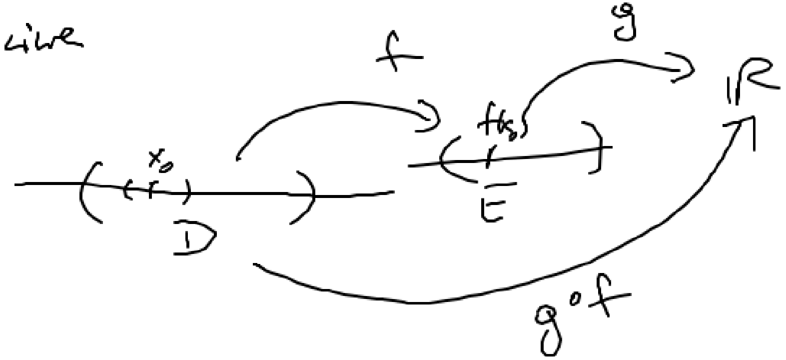
$$(\cos x)' = -\sin x$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

$$\|$$

$$-1 - \operatorname{ctg}^2 x$$

Tw. Załóżmy, że $f: D \to E$, $g: E \to \mathbb{R}$, $(x_0-\eta, x_0+\eta) \subset D$, $(f(x_0)-\eta, f(x_0)+\eta) \subset E$ dla pewnego $\eta > 0$, oraz że istnieją pochodne właściwe



$$f'(x_0) \quad \text{oraz} \quad g'(f(x_0)).$$

Wówczas $g \circ f$ ma pochodną w x_0 oraz

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0).$$

Np.

$$\Big(\underbrace{\sin}_{g}(\underbrace{\cos}_{f}(x))\Big)' = \big((g \circ f)(x)\big)' = g'(f(x)) \cdot f'(x) = \cos(f(x))(-\sin x) =$$

$$= \cos(\cos(x)) \cdot (-\sin x)$$

$g = \sin$ $\qquad g(y) = \sin y \to g'(y) = \cos y$

$f = \cos$ $\qquad f(x) = \cos x \to f'(x) = -\sin x$

Np.

$$\left((x^2+1)^3\right)' = g'\big(\underbrace{f(x)}_{y}\big)\cdot f'(x) = 3\big(f(x)\big)^2\cdot 2x = 3\left(x^2+1\right)^2\cdot 2x$$

$$(x^2+1)^3$$

$$\begin{tikzcd} x \arrow[r, "f(x)=x^2+1"] \arrow[rr, bend right, "(g\circ f)(x)"'] & x^2+1 \arrow[r, "g(y)=y^3"] & (x^2+1)^3 \end{tikzcd}$$

$$g(y) = y^3$$

$$g'(y) = 3\cdot y^{3-1} = 3y^2$$

$$f(x) = x^2+1$$

$$f'(x) = 2x+0 = 2x$$

Inaczej:

$$\left((x^2+1)^3\right)' = \left(x^6+3x^4+3x^2+1\right)' = 6x^5+12x^3+6x+0$$

$$\left((x^2+1)^3\right)' = \left((x^2+1)^2\cdot(x^2+1)\right)' = \ldots$$

Nf:

$$\left(e^{-x^2}\right)' = g'(f(x)) \cdot f'(x) = e^{-x^2} \cdot (-2x)$$

$$\begin{tikzcd}
x \arrow[r, "f(x) = -x^2"] \arrow[rr, bend right, "(g \circ f)(x)"'] & \underbrace{-x^2}_{= y} \arrow[r, "g(y) = e^y"] & e^{-x^2}
\end{tikzcd}$$

$$f(x) = -x^2 = (-1) \cdot x^2$$

$$f'(x) = -1 \cdot (x^2)' = -2x$$

$$g(y) = e^y$$

$$g'(y) = e^y$$

Np:

$$\left(\sin^2(x)\right)' = g'(\underbrace{\sin x}_{y}) \cdot f'(x) = 2\sin x \cdot \cos x$$

$$\begin{tikzcd}
x \arrow[r, bend left, "f(x)=\sin x"] \arrow[r, bend right, "(g\circ f)(x)"'] & \sin(x) = y \arrow[r, bend left, "g(y)=y^2"] & \sin^2 x
\end{tikzcd}$$

$$\sin^2 x = (\sin(x))^2$$

$$f(x) = \sin x$$

$$f'(x) = \cos x$$

$$g(y) = y^2 \qquad g'(y) = 2y$$

Inaczej:

$$(\sin^2 x)' = (\sin x \cdot \sin x)' =$$

$$= (\sin x)' \cdot \sin x + \sin x\,(\sin x)' =$$

$$= \cos x \sin x + \sin x \cos x = 2\sin x \cos x$$