$$\frac{P(x)}{Q(x)} = f(x) = I(x) + \frac{R(x)}{Q(x)}$$

całk. sprowadza się do całk. f. wymiernych ułamków prostych I rodzaju:

$$\frac{A}{(x+a)^n} \quad n \in \mathbb{N}$$

$$\int \frac{A\,dx}{(x+a)^n} = \begin{cases} A \ln|x+a| & \text{dla } n = 1 \\ \frac{A}{-n+1}(x+a)^{-n+1} = -\frac{A}{n-1} \cdot \frac{1}{(x+a)^{n-1}} & n \geq 2 \end{cases}$$

Ułamki proste II rodzaju;

$$\frac{Ax+B}{(x^2+px+q)^n} \qquad p^2-4q<0 \qquad n\in \mathbb{N}$$

Tw. Każda funkcja wymierna właściwa da się przedstawić jako suma ułamków prostych I lub II rodzaju.

Przykłady rozkładów ogólnych:

$$\frac{x^4+1}{(x-2)^3 x^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{x} + \frac{E}{x^2}$$

$$\frac{x^3+2x+5}{(x^2+2x+6)^2(x^2+1)^2(x-3)^3} = \frac{Ax+B}{x^2+2x+6} + \frac{Cx+D}{(x^2+2x+6)^2}$$

$$+ \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{J}{x-3} + \frac{K}{(x-3)^2} + \frac{L}{(x-3)^3}$$

$$\int \frac{2\,dx}{3x+5} = \frac{2}{3}\int \frac{dx}{x+\frac{5}{3}} = \frac{2}{3}\ln\left|x+\frac{5}{3}\right| + C$$

$$\int \frac{2\,dx}{(3x+5)^2} = \frac{2}{9}\int \frac{dx}{\left(x+\frac{5}{3}\right)^2} = -\frac{2}{9}\cdot\frac{1}{x+\frac{5}{3}} + C$$

$$= -\frac{2}{3}\cdot\frac{1}{3x+5} + C$$

$$\int \frac{dx}{x^2+x-2} = \int \frac{dx}{(x-1)(x+2)}$$

$\Delta = 1+8 = 9$

$\sqrt{\Delta} = 3$

$x_1 = \frac{-1-3}{2} = -2$

$x_2 = \frac{-1+3}{2} = 1$

$$= \frac{1}{3} \int \left(\frac{1}{x-1} - \frac{1}{x+2} \right) dx =$$

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2} \quad \Big/ (x-1)(x+2)$$

$$1 = A(x+2) + B(x-1)$$

$x = 1 \qquad 1 = 3A \qquad A = \frac{1}{3}$

$x = -2 \qquad 1 = -3B \qquad B = -\frac{1}{3}$

$$= \frac{1}{3} \ln|x-1| - \frac{1}{3} \ln|x+2| + C$$

$$= \frac{1}{3} \ln\left|\frac{x-1}{x+2}\right| + C$$

$$\int \frac{dx}{(x+1)(x-2)(x-3)} = \int \left(\frac{1}{12} \cdot \frac{1}{x+1} - \frac{1}{3} \cdot \frac{1}{x-2} + \frac{1}{4} \cdot \frac{1}{x-3} \right) dx$$

$$= \frac{1}{12} \ln|x+1| - \frac{1}{3} \ln|x-2| + \frac{1}{4} \ln|x-3| + C =$$

$$= \frac{1}{12} \ln \left| \frac{(x+1)(x-3)^3}{(x-2)^4} \right| + C$$

$$\frac{1}{(x+1)(x-2)(x-3)} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{x-3} \quad \Big| \cdot M$$

$$1 = A(x-2)(x-3) + B(x+1)(x-3) + C(x+1)(x-2)$$

$x = -1 \qquad 1 = 12A \qquad A = \frac{1}{12} \quad B = -\frac{1}{3} \quad C = \frac{1}{4}$

$x = 2 \qquad 1 = -3B$

$x = 3 \qquad 1 = 4C$

$$\int \frac{x^4+1}{x^2+x-2}\,dx = \int \left(x^2 - x + 3 + \frac{-5x+7}{(x+1)(x-2)} \right) dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 3x - \int \left(\frac{4}{x+1} + \frac{1}{x-2} \right) dx =$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + 3x - 4\ln|x+1| - \ln|x-2| + C$$

$$\frac{-5x+7}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \quad \Big| (x+1)(x-2)$$

$$-5x+7 = A(x-2) + B(x+1)$$

$x = -1$ $\quad 12 = -3A$

$x = 2$ $\quad -3 = 3B$

$$A = -4, \; B = -1$$

$$\int \frac{Ax+B}{x^2+px+q}\,dx$$

$p^2-4q<0$

$u'=2x+p$

$$\frac{Ax+B}{x^2+px+q} = \frac{\frac{A}{2}(2x+p)-\frac{Ap}{2}+B}{x^2+px+q} =$$

$$= \frac{A}{2}\cdot\frac{2x+p}{x^2+px+q} + \left(B-\frac{Ap}{2}\right)\cdot\frac{1}{x^2+px+q}$$

$$\int \frac{dx}{x^2+px+q} \overset{p^2-4q<0}{=} \int \frac{dx}{x^2+2\cdot\frac{p}{2}x+\frac{p^2}{4}-\frac{p^2}{4}+q} =$$

$$= \int \frac{dx}{\left(x+\frac{p}{2}\right)^2+\underbrace{\frac{4q-p^2}{4}}_{>0}} = \left| \begin{array}{l} t = x+\frac{p}{2} \\ dt = dx \\ a \overset{def}{=} \sqrt{\frac{4q-p^2}{4}} \end{array} \right| =$$

$$= \int \frac{dt}{t^2+a^2} = \frac{1}{a} \operatorname{arctg} \frac{t}{a} + C$$

$t = au$ $\quad dt = a\,du,$ $\quad u = \frac{t}{a}$

$$\int \frac{a\,du}{a^2u^2+a^2} = \frac{a}{a^2}\int \frac{du}{u^2+1}$$

$$\frac{1}{a} \operatorname{arctg} u + C$$

Precvičme,

$$\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{x^2+4x+4+9} =$$

$$= \int \frac{dx}{(x+2)^2+9} \underset{a=3}{=} \frac{1}{3}\operatorname{arctg}\frac{x+2}{3}+C$$

$t=x+2$

II Sp.

$$\frac{1}{9}\int \frac{dx}{\left(\frac{x+3}{3}\right)^2+1} = \left| \begin{array}{l} t=\frac{x+3}{3} \\ 3t=x+2 \\ dx=3dt \end{array} \right| = \frac{1}{9}\int \frac{3dt}{t^2+1}$$

$$= \frac{1}{3}\operatorname{arctg} t + C$$

$$\int \frac{Ax+B}{(x^2+px+q)^n}\,dx = \int \frac{Ax+B}{\left(\left(x+\frac{p}{2}\right)^2+\frac{-\Delta}{4}\right)^n}$$

Problem sprowadza się do liczenia całek postaci:

$$t = \frac{x+\frac{p}{2}}{\frac{-\Delta}{4}}$$

$$\int \frac{dx}{(x^2+1)^n}$$

$$\int \frac{dx}{x^4+2x^3+5x^2} = \int \frac{dx}{x^2(x^2+2x+5)}$$

$$\frac{1}{x^2(x^2+2x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+5} \quad / x^2(x^2+2x+5)$$

$$1 = Ax(x^2+2x+5) + B(x^2+2x+5) + (Cx+D)x^2$$

$x=0$ $\quad 1 = 5B \quad B=\frac{1}{5}$

$x=1$ $\quad 1 = 8A + 8B + C + D$

$x=-1$ $\quad 1 = -4A + 4B - C + D$

$x=-2$ $\quad 1 = -10A + 5B - 8C + 4D$

$6D - 2C = -\frac{3}{5}$

$$\begin{cases} 8A + C + D = -\frac{3}{5} \\ -4A - C + D = \frac{1}{5} \\ -10A - 8C + 4D = 0 \end{cases}$$

$4A + 2D = -\frac{2}{5} \quad | \cdot 3$

$12A + 2C = -\frac{4}{5}$

$$\begin{cases} 2A+D=-\frac{1}{5} \\ 3D-C=-\frac{1}{5} \\ -4C+2D=5A \end{cases}$$

$$A=-\frac{1}{10}-\frac{D}{2}$$

$$\begin{cases} 3D-C=-\frac{1}{5} \quad | \cdot(-4) \\ -4C+2D=-\frac{1}{2}-\frac{5}{2}D \end{cases}$$

$$B=\frac{1}{5}$$

$$C=3D+\frac{1}{5}=\frac{2}{25}$$

$$-4C+\frac{9}{2}D=-\frac{1}{2}$$

$$4C-12D=\frac{4}{5}$$

$$A=-\frac{1}{10}+\frac{1}{50}=-\frac{4}{50}=-\frac{2}{25}$$

$$-\frac{15}{2}D=\frac{3}{10}$$

$$D=\frac{\cancel{3}^{1}}{\cancel{10}_{5}}\cdot\frac{\cancel{2}^{1}}{\cancel{15}_{5}}=-\frac{1}{25}$$

$$\begin{cases} A=-\frac{2}{25} \\ B=\frac{1}{5} \\ C=\frac{2}{25} \\ D=-\frac{1}{25} \end{cases}$$

$$\int\left(-\frac{2}{25}\cdot\frac{1}{x}+\frac{1}{5}\cdot\frac{1}{x^2}+\frac{\frac{2}{25}x-\frac{1}{25}}{x^2+2x+5}\right)=$$

$$=-\frac{2}{25}\ln|x|-\frac{1}{5}\cdot\frac{1}{x}+\frac{1}{25}\int\frac{2x+2-3}{x^2+2x+5}\,dx=$$

$$=-\frac{2}{25}\ln|x|-\frac{1}{5}\cdot\frac{1}{x}+\frac{1}{25}\ln\left(x^2+2x+5\right)\underbrace{-\frac{3}{25}\int\frac{dx}{x^2+2x+5}}_{-\frac{3}{50}\cdot\operatorname{arctg}\frac{x+1}{2}}$$

$$\int\frac{dx}{x^2+2x+5}=\int\frac{dx}{(x+1)^2+2^2}=\frac{1}{2}\operatorname{arctg}\frac{x+1}{2}+C$$

$+C$

$$-\frac{2}{25} \cdot \frac{1}{x}$$

$$\int_0^\pi x^3 \sin x\,dx = \left|\begin{matrix} f = x^3 & g = -\cos x \\ f' = 3x^2 & g' = \sin x \end{matrix}\right| =$$

$$= -x^3 \cos x \Big|_0^\pi + 3\int_0^\pi x^2 \cos x\,dx = \left|\begin{matrix} f = x^2 & g = \sin x \\ f' = 2x & g' = \cos x \end{matrix}\right|$$

$$= \pi^3 + 3\left(x^2 \sin x \Big|_0^\pi - 2\int_0^\pi x \sin x\,dx \right) = \left|\begin{matrix} f = x & g = -\cos x \\ f' = 1 & g' = \sin x \end{matrix}\right|$$

$$= \pi^3 + 3\left(0 - 2\left(-x\cos x \Big|_0^\pi + \int_0^\pi \cos x\,dx\right)\right) = \pi^3 - 6\left(\pi + \sin x \Big|_0^\pi\right)$$

$$= \underline{\pi^3 - 6\pi}$$