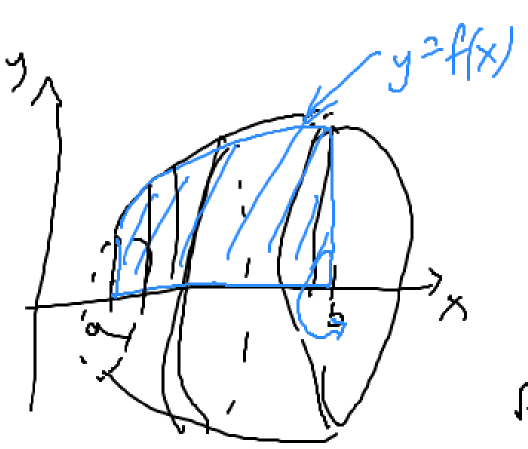
~~obj. i~~ pole

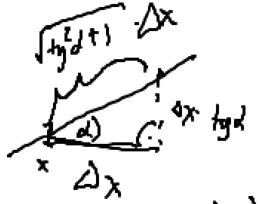
zał. $f:[a,b] \to [0,\infty)$

Obszar $\{(x,y,0):\ x \in [a,b],\ 0 \le y \le f(x)\}$

obracamy wokół $Ox$ w przestrzeni

obj. powstałej bryły $= \int_a^b \pi f^2(x)\, dx$

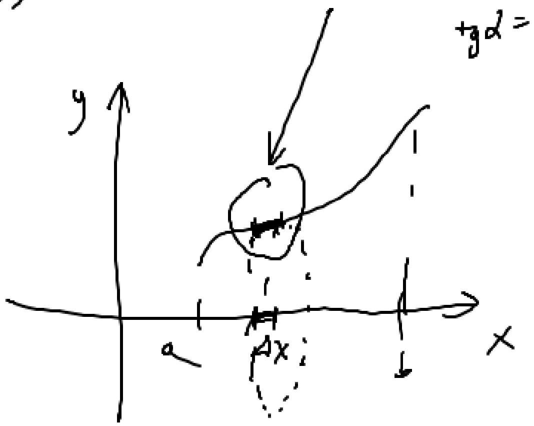


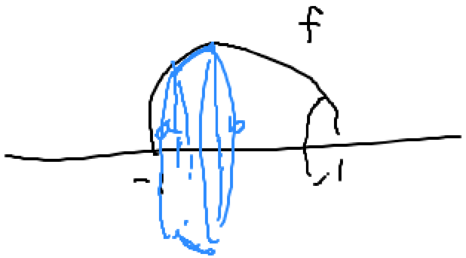

$tg\alpha = f'(x)$

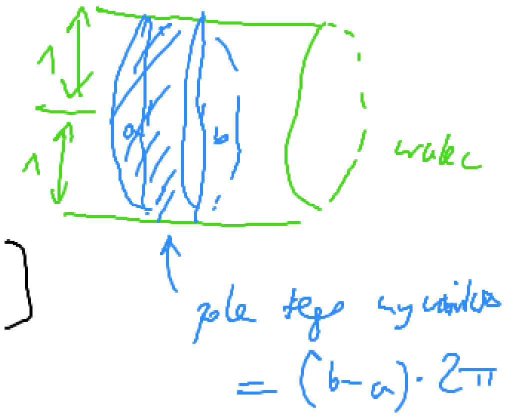
pole powierzchni otrzymanej przez obrót $\{(x, f(x), 0) : x \in [a,b]\}$

$$= 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx$$



$f(x) = y = \sqrt{1-x^2}$ , $x \in [-1,1]$ półokrąg




Obliczymy pole wycinka sfery powstałego przez obrót

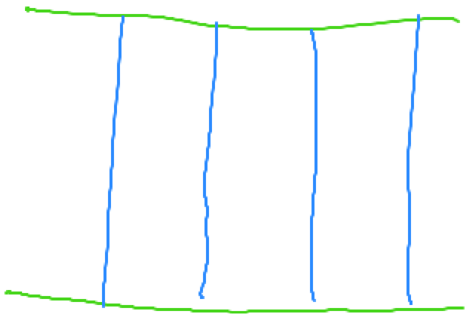
$$\{(x, f(x), 0) : x \in [a,b]\}$$

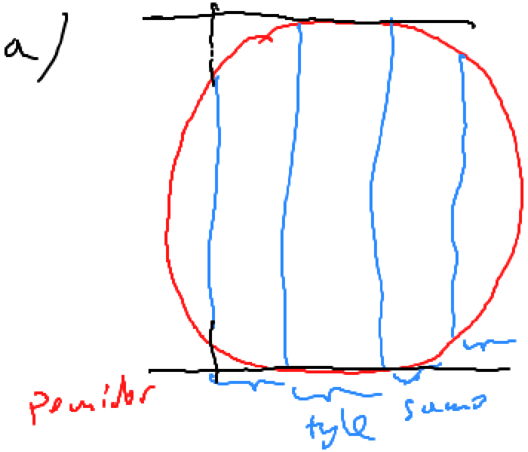
wokół $Ox$.

$$[a,b] \subset [-1,1]$$

$$f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}} \qquad 1 + (f'(x))^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$P = 2\pi \int_a^b \underbrace{\sqrt{1-x^2}}_{f(x)} \cdot \underbrace{\frac{1}{\sqrt{1-x^2}}}_{\sqrt{1+(f'(x))^2}} dx = 2\pi \int_a^b dx = 2\pi \, x\Big|_a^b = 2\pi(b-a)$$





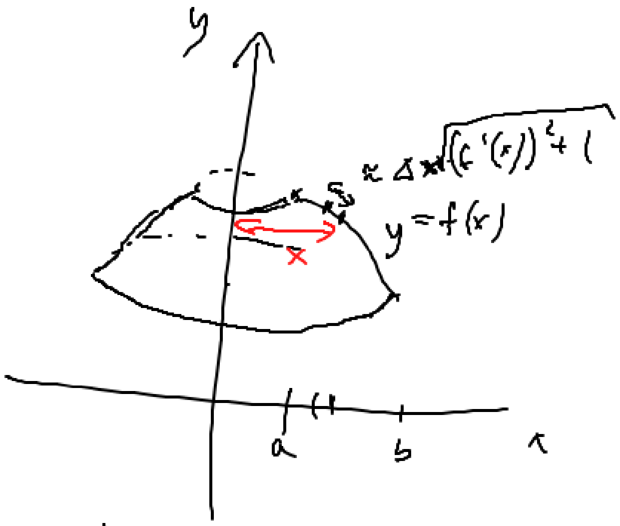

O tym wiedział już Archimedes.

Pole powierzchni powstałej przez obrót

$\{(x, f(x), 0) : x \in [a,b]\}$ (gdy $[a,b] \subset [0,\infty)$) ~~lub [a,b]~~ $\subset$

wokół $Oy$

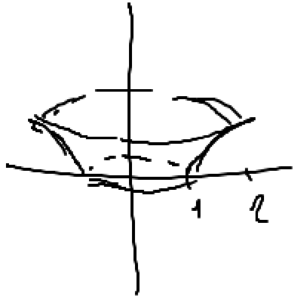
$$= 2\pi \int_a^b x\sqrt{1+(f'(x))^2}\,dx$$




Np. $f(x) = \ln x, \quad x \in [1,2] \qquad f'(x) = \frac{1}{x}$

$$P = 2\pi \int_1^2 x\sqrt{1+\frac{1}{x^2}}\,dx =$$

$$= 2\pi \int_1^2 x\,\frac{\sqrt{x^2+1}}{\underbrace{|x|}_{=x,\text{ bo } x\in[1,2]}}\,dx = 2\pi \int_1^2 \sqrt{1+x^2}\,dx = ?$$



№1.

$x^2$

$$\int \sqrt{1-x^2}\,dx = \left| \begin{array}{l} x = \sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad t = \arcsin x \\ dx = \cos t\,dt \\ \sqrt{1-x^2} = \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = \cos t \end{array} \right| =$$

$$\begin{cases} \cos 2t = \cos^2 t - \sin^2 t \\ 1 = \cos^2 t + \sin^2 t \end{cases}$$

$$+: \quad 1 + \cos 2t = 2\cos^2 t$$

$\sin^2 t$

$$= \int \cos t \cdot \cos t\,dt = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2t\right) dt = \frac{1}{2} t + \frac{1}{2}\cdot\frac{1}{2}\sin 2t + C =$$

$$= \frac{1}{2} t + \frac{1}{4}\sin 2t + C = \frac{1}{2}\arcsin x + \frac{1}{2}\sin t \cos t + C =$$

$$= \underline{\frac{1}{2}\arcsin x + \frac{1}{2} x\sqrt{1-x^2} + C}$$


№4.

$$P_{\frown} = \int_{-1}^{1} \sqrt{1-x^2}\,dx = \left(\frac{1}{2}\arcsin x + \frac{1}{2} x\sqrt{1-x^2}\right)\Big|_{-1}^{1} = \frac{1}{2}\arcsin 1 - \frac{1}{2}\arcsin(-1) =$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}$$

Np. $\int \sqrt{x^2+1}\,dx = \left| \begin{array}{l} x = \operatorname{sh} t,\ t \in \mathbb{R} \\ \sqrt{1+x^2} = \sqrt{1+\operatorname{sh}^2 t} = \sqrt{\operatorname{ch}^2 t} = \operatorname{ch} t \\ dx = \operatorname{ch} t\, dt \end{array} \right| =$

$\operatorname{sh} x = \frac{e^x - e^{-x}}{2} \qquad (\operatorname{sh} x)' = \operatorname{ch} x$

$\operatorname{ch} x = \frac{e^x + e^{-x}}{2} \geq 0 \qquad (\operatorname{ch} x)' = \operatorname{sh} x$

$= \int \operatorname{ch} t \cdot \operatorname{ch} t\, dt = \int \operatorname{ch}^2 t\, dt =$

$\operatorname{ch}^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4}$

$= \int \left(\frac{e^t + e^{-t}}{2}\right)^2 dt = \int \frac{e^{2t} + 2 + e^{-2t}}{4}\, dt =$

$\operatorname{sh}^2 x = \frac{e^{2x} - 2 + e^{-2x}}{4}$

$= \frac{1}{4}\left(\frac{e^{2t}}{2} + 2t - \frac{e^{-2t}}{2}\right) + C \quad = ?$

$\Downarrow$

$$\boxed{\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1}$$

$\operatorname{ch}^2 x = 1 + \operatorname{sh}^2 x$

Ale $\int_1^2 \sqrt{x^2+1} = \int_{\operatorname{sh} 1}^{\operatorname{sh} 2} \operatorname{ch}^2 t\, dt = \ldots = \frac{1}{4}\left(\frac{e^{2t}}{2} + 2t - \frac{e^{-2t}}{2}\right)\Big|_{\operatorname{sh} 1}^{\operatorname{sh} 2}$

Cd. $\begin{cases} x = \operatorname{sh} t \end{cases}$

$e^{\ln a} = a$

$e^{2\ln a} = \left(e^{\ln a}\right)^2 = a^2$

$\operatorname{sh} t = \frac{e^t - e^{-t}}{2}$

$\operatorname{ch} t = \frac{e^t + e^{-t}}{2}$

$$\ldots = \frac{1}{4}\left(\frac{e^{2t}}{2} + 2t - \frac{e^{-2t}}{2}\right) + C =$$

$$= \frac{1}{4}\left(\frac{e^{2\ln(\sqrt{1+x^2}+x)}}{2} + 2\ln\left(\sqrt{1+x^2}+x\right) - \frac{e^{-2\ln(\sqrt{1+x^2}+x)}}{2}\right) + C =$$

$$= \frac{1}{8}\left(\sqrt{1+x^2}+x\right)^2 + \frac{1}{2}\ln\left(\sqrt{1+x^2}+x\right) - \frac{1}{8}\,\frac{1}{\left(\sqrt{1+x^2}+x\right)^2} + C$$

$\operatorname{ch} t + \operatorname{sh} t = e^t$

$\operatorname{ch} t = \sqrt{1+\operatorname{sh}^2 t}$

$\sqrt{1+\operatorname{sh}^2 t} + \operatorname{sh} t = e^t \quad / \ln(\,)$

$t = \ln\left(\sqrt{1+\operatorname{sh}^2 t} + \operatorname{sh} t\right)$

$= \ln\left(\sqrt{1+x^2} + x\right)$

№1.

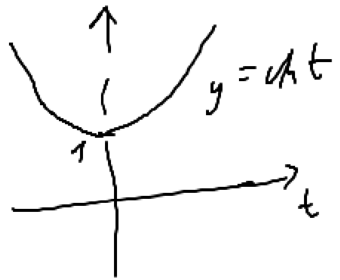
$$\int \sqrt{x^2-1}\,dx = \left| \begin{array}{l} x = \operatorname{ch} t,\ t \geq 0 \\ \sqrt{x^2-1} = \sqrt{\operatorname{sh}^2 t} = \operatorname{sh} t \\ dx = \operatorname{sh} t\,dt \end{array} \right| =$$

$$= \int \operatorname{sh}^2 t\,dt = \int \left(\frac{e^t - e^{-t}}{2}\right)^2 dt = \int \frac{e^{2t} - 2 + e^{-2t}}{4}\,dt =$$

$$= \frac{1}{8}e^{2t} - \frac{1}{2}t - \frac{1}{8}e^{-2t} + C =$$

$$= \frac{1}{8}e^{2\ln\left(x+\sqrt{x^2-1}\right)} - \frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right) - \frac{1}{8}e^{-2\ln\left(x+\sqrt{x^2-1}\right)} + C$$

$$\operatorname{ch}^2 t - 1 = \operatorname{sh}^2 t$$




$$\operatorname{ch} t = \frac{e^t + e^{-t}}{2}$$

$$\operatorname{sh} t = \frac{e^t - e^{-t}}{2}$$

$$\operatorname{ch} t + \sqrt{\operatorname{ch}^2 t - 1} = \operatorname{ch} t + \operatorname{sh} t = e^t \quad \Big/ \ln(\cdot)$$

$$t = \ln\left(\operatorname{ch} t + \sqrt{\operatorname{ch}^2 t - 1}\right), \quad t \geq 0$$

Р. домашнее для гр.

$$\int \frac{x^2\sqrt{x^2-1}}{1+x}\,dx$$

Np.

$$\int \sqrt{1+x-x^2}\,dx = \int \sqrt{-(x^2-x-1)}\,dx = \int \sqrt{-\left(x-\frac{1}{2}\right)^2+\frac{5}{4}}\,dx =$$

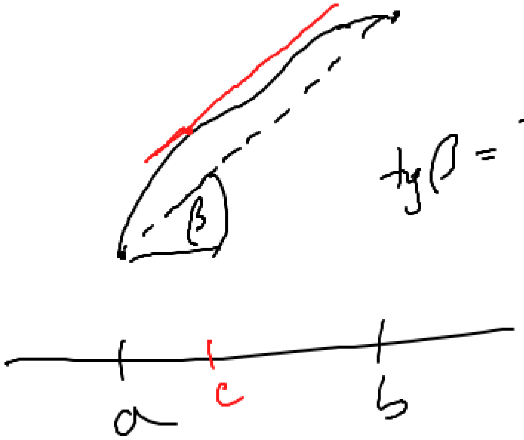
$$= \sqrt{\frac{5}{4}} \int \sqrt{1-\frac{4}{5}\left(x-\frac{1}{2}\right)^2}\,dx = \sqrt{\frac{5}{4}} \int \sqrt{1-\left(\frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right)\right)^2}\,dx =$$

$$= \left| \begin{array}{l} \frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right) = \sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \frac{2}{\sqrt{5}}dx = \cos t\, dt \\ \sqrt{1-\sin^2 t} = |\cos t| = \cos t \\ t = \arcsin\left(\frac{2}{\sqrt{5}}\left(x-\frac{1}{2}\right)\right) \end{array} \right| = \sqrt{\frac{5}{4}} \int \cos^2 t\, dt \cdot \frac{\sqrt{5}}{2} = \ldots$$

## Tw. Lagrange'a

Jeśli $f: [a,b] \to \mathbb{R}$ cg., różniczkowalna na $(a,b)$, to istnieje $c \in (a,b)$:

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



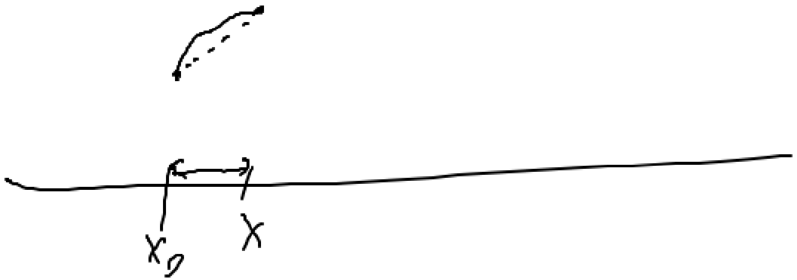

$$\operatorname{tg}\beta = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \quad f(b) - f(a) = f'(c)\cdot(b-a)$$

$$f(b) = f(a) + f'(c)\cdot(b-a)$$

$b = x$, $a = x_0$

$$f(x) = f(x_0) + f'(c)\cdot(x - x_0)$$



np.

$$f(x) = \sqrt{1+x} \qquad f'(x) = \frac{1}{2\sqrt{1+x}}\,(1+x)' = \frac{1}{2\sqrt{1+x}} \cdot 1$$

$$x_0 = 0$$

$$\underbrace{\sqrt{1+x}}_{f(x)} = \underbrace{1}_{f(x_0)} + \frac{1}{2\sqrt{1+c}} \cdot \underbrace{x}_{x-x_0}$$

dla pewnego $c \in (0, x)$ $\quad (x > 0)$

czyli np.

$$\sqrt{1{,}03} = 1 + \frac{1}{2\sqrt{1+c}} \cdot 0{,}03 \in \qquad \text{-}\|\text{-} \quad c \in (0,\ 0.03)$$

$$\in \left[1 + \frac{1}{2\sqrt{1.03}}\, 0.03\ ,\ 1 + \frac{1}{2} \cdot 0.03\right] \subset \left(1 + \frac{1}{2{,}2} \cdot 0.03\ ,\ 1{,}015\right) \subset$$

$$\subset (1.01\ ,\ 1.015)$$

$$1.03 < 1.21 = (1.1)^2$$

Uogólnienie:

**Wzór Taylora:**

Załóżmy, że $f$ ma pochodną do rzędu $n$ włącznie w otoczeniu punktu $x_0$, powiedzmy w $(x_0-\delta, x_0+\delta)$.
Wtedy dla dowolnego $x \in (x_0-\delta, x_0+\delta)$ istnieje $c$ pomiędzy $x_0$ a $x$ (czyli $c \in (x_0, x)$ gdy $x_0 < x$, $c \in (x, x_0)$ gdy $x < x_0$,
$c = x_0$ gdy $x = x_0$) takie, że


$$f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}}_{\text{wzór przybliżony}} + \underbrace{\frac{f^{(n)}(c)}{n!}(x-x_0)^n}_{=: R_n(x) \text{ – reszta}}$$

N1.

$f(x) = \sin x \qquad x_0 = 0 \qquad n = 5$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{IV}(x) = \sin x$

$f^{V}(x) = \cos x$

$$\sin x = 0 + x + 0 + (-1)\frac{x^3}{6} + 0 + \cos c \cdot \frac{x^5}{5!}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + f^{IV}(x_0)\frac{(x-x_0)^4}{4!} + R_5(c)$$

$$\Rightarrow \sin x = x - \frac{x^3}{6} + \cos c \cdot \frac{x^5}{120}$$ , gdzie c - pomiędzy 0 a x

Np. dla $|x| < 0,1$ :

$$\sin x \approx x - \frac{x^3}{6}$$

$$|\text{błąd oszacl.}| \le |R_5(c)| = |\cos c| \cdot \left|\frac{x^5}{120}\right| \le 1 \cdot \frac{10^{-5}}{120} < 10^{-7}$$

Jak policzyć $\sin(10)$ ?

$$\sin(10) = -\sin(10 - 3\pi)$$