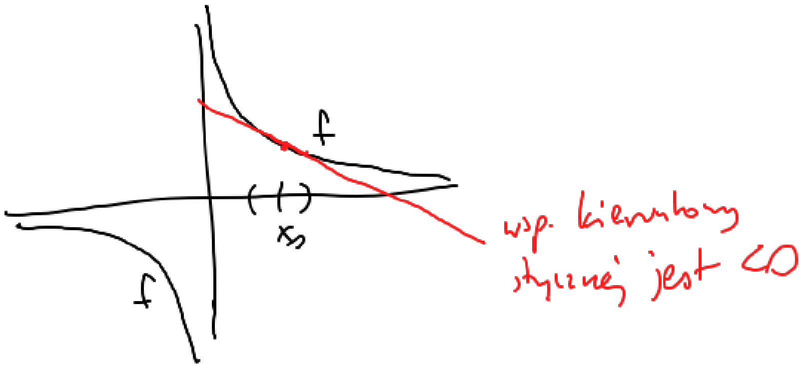
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$



Np.

$$f(x) = \frac{1}{x}, \quad x \in \mathbb{R} \setminus \{0\}$$

Niech $x_0 \in \mathbb{R} \setminus \{0\}$.

$$f'(x_0) = \lim_{h \to 0} \frac{\frac{1}{x_0+h} - \frac{1}{x_0}}{h} = \lim_{h \to 0} \frac{\cancel{x_0} - (\cancel{x_0} + \cancel{h})}{\cancel{h}\, x_0 (x_0+h)} = \lim_{h \to 0} \frac{-1}{x_0 \underbrace{(x_0+h)}_{\downarrow \, x_0}} = -\frac{1}{x_0^2}$$

Tw. Jeśli f, g są różniczkowalne w punkcie x_0, to funkcje $f+g$, $f-g$, fg też są różniczkowalne w x_0. Jeśli dodatkowo $g(x_0) \neq 0$, to też $\frac{f}{g}$ jest różniczkowalna.

Mamy

$$(f+g)'(x_0) = f'(x_0) + g'(x_0)$$

$$(f-g)'(x_0) = f'(x_0) - g'(x_0)$$

$$(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$$

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) g(x_0) - f(x_0) g'(x_0)}{g^2(x_0)}$$

o ile $g(x_0) \neq 0$.

Dow dla fg:

$$(fg)'(x_0) = \lim_{h\to 0} \frac{(fg)(x_0+h) - (fg)(x_0)}{h} =$$

$$= \lim_{h\to 0} \frac{f(x_0+h)\cdot g(x_0+h) - f(x_0)\,g(x_0+h) + f(x_0)\,g(x_0+h) - f(x_0)\,g(x_0)}{h} =$$

$$= \lim_{h\to 0}\left(\underbrace{g(x_0+h)}_{\xrightarrow{h\to 0}\, g(x_0)} \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{\downarrow\, f'(x_0)} + f(x_0) \underbrace{\frac{g(x_0+h) - g(x_0)}{h}}_{\downarrow\, g'(x_0)} \right) = g(x_0)\,f'(x_0) + f(x_0)\,g'(x_0)$$

(pokażemy za chwilę)

$\otimes$

Uzupełnienie dowodu:

<u>Tw.</u> Jeśli g jest różniczkowalna w x_0, to g jest ciągła w x_0.

<u>D-d.</u>

$$g'(x_0) = \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h}$$

Weźmy dowolny ciąg (h_n), $h_n \to 0$, $h_n \neq 0$. Z powyższej równości wynika, że

$$\lim_{n\to\infty} \frac{g(x_0+h_n) - g(x_0)}{h_n} = g'(x_0).$$

Weźmy $\varepsilon = 1$ i skorzystajmy z def. granicy ciągu: istnieje n_0 takie, że dla $n \geq n_0$ zachodzi

$$\left| \frac{g(x_0+h_n) - g(x_0)}{h_n} - g'(x_0) \right| < \varepsilon \qquad \Big| \cdot |h_n|$$

stąd $\quad |g(x_0+h_n) - g(x_0) - g'(x_0)\cdot h_n| < \varepsilon |h_n|$

$$\overset{\overset{0}{\uparrow}}{0} \leq |g(x_0+h_n) - g(x_0)| \underset{\text{nier. trójkąta}}{\leq} |g(x_0+h_n) - g(x_0) - g'(x_0)h_n| + |g'(x_0)|\cdot|h_n| < \varepsilon\cdot|h_n| + |g'(x_0)|\cdot|h_n| = \underbrace{(1+|g'(x_0)|)\cdot|h_n|}_{\xrightarrow{n\to\infty} 0}$$

z tw. o 3 ciągach $|g(x_0+h_n) - g(x_0)| \to 0$, stąd $\lim_{h\to 0} g(x_0+h) = g(x_0)$ $\Big|$ 0

• $(x^n)' = n x^{n-1}$

• dla $n \in \mathbb{Z}$ i $x \in \mathbb{R} \setminus \{0\}$ — dla $n=-1$: $\left(\frac{1}{x}\right)' = -\frac{1}{x^2} = -x^{-2}$, gdzie $\frac{1}{x} = x^{-1}$

• dla $n \in \mathbb{R}$ i $x > 0$

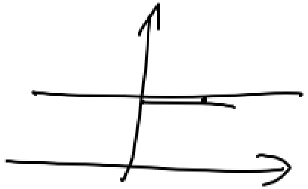
• dla $n \in \mathbb{N}$ i $x \in \mathbb{R}$

Konwencja: $0^0 = 1$

$(x)' = 1$, $x \in \mathbb{R}$

$(1)' = 0$, $x \in \mathbb{R}$ $\rightsquigarrow$ jeśli $f(x) = 1$ dla $x \in \mathbb{R}$, to $f'(x) = 0$ dla $x \in \mathbb{R}$

• Jeśli $f(x) = c$ dla $x \in \mathbb{R}$ (funkcja stała), to



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = \underline{0}$$

$(fg)' = f'g + fg'$ $\qquad$ $(x^n)' = nx^{n-1}$

Nr. 1.

$$\left(2x^7 + x^3 + 4\right)' = \left(2x^7 + x^3\right)' + (4)' = (2x^7)' + (x^3)' + \underset{\text{f. stała}}{(4)'} =$$

$$= \underset{\text{f. stała}}{(2)'} \cdot x^7 + 2 \cdot (x^7)' + 3x^{3-1} + 0 =$$

$$= 0 \cdot x^7 + 2 \cdot 7x^{7-1} + 3x^2 = \underline{14x^6 + 3x^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Nr. I

$$\left(\frac{\sqrt{x}+1}{x}\right)' = \frac{(\sqrt{x}+1)' \cdot x - (\sqrt{x}+1) \cdot (x)'}{x^2} = \frac{\left((x^{\frac{1}{2}})' + (1)'\right) \cdot x - (\sqrt{x}+1) \cdot 1}{x^2} =$$

$$= \frac{\left(\frac{1}{2}x^{\frac{1}{2}-1} + 0\right)x - \sqrt{x} - 1}{x^2} = \frac{\frac{1}{2}\sqrt{x} - \sqrt{x} - 1}{x^2} = \frac{-\frac{1}{2}\sqrt{x} - 1}{x^2}$$

II $\left(\frac{\sqrt{x}+1}{x}\right)' = \left(\frac{\sqrt{x}}{x}+\frac{1}{x}\right)' = \left(x^{-\frac{1}{2}}\right)' + \left(x^{-1}\right)' =$

$(x^n)' = n x^{n-1}$

$= -\frac{1}{2} x^{-\frac{1}{2}-1} + (-1) x^{-1-1} = -\frac{1}{2} x^{-\frac{3}{2}} - x^{-2}$

Fakt.

$(e^x)' = e^x$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h-1)(\cos h+1)}{h(\cos h+1)} =$

$= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h+1)} = \lim_{h\to 0} \underbrace{\frac{\sin h}{h}}_{\downarrow\, 1} \cdot \underbrace{\frac{(\overbrace{-\sin h}^{\to 0})}{\cos h+1}}_{\downarrow\, 2} = 0$

Mp. $(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} =$

$= \lim_{h\to 0} \left(\sin x \underbrace{\frac{\cos h - 1}{h}}_{\downarrow\, 0}\right) + \lim_{h\to 0} \cos x \underbrace{\frac{\sin h}{h}}_{\downarrow\, 1} = \cos x.$

$(fg)' = f'g + fg'$

$(e^x)' = e^x$

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Пр. $(e^x \cdot \sin^2 x)' = (e^x)' \cdot \sin^2 x + e^x \cdot (\sin^2 x)' =$

$= e^x \sin^2 x + e^x (\sin x \cdot \sin x)' =$

$= e^x \sin^2 x + e^x \left(\underbrace{(\sin x)'}_{\cos x} \sin x + \sin x \underbrace{(\sin x)'}_{\cos x} \right) =$

$= \underline{e^x \sin^2 x + e^x \cdot 2 \cos x \sin x}$

Пр. $(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cdot \cos x - \sin x \cdot (\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x$

Np.

$$(\operatorname{ctg} x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \cdot \sin x - \cos x\,(\sin x)'}{\sin^2 x} =$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

$$= -1 - \operatorname{ctg}^2 x$$

$$(\cos x)' = -\sin x$$

$$(\sin x)' = \cos x$$

Fakt. Jeśli $f'(x_0) \neq 0$ i f ma funkcję odwrotną (w otoczeniu x_0), to

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$$

Np. $f(x) = e^x \rightsquigarrow f^{-1}(x) = \ln x, \quad x > 0$

$$(f^{-1})'(e^{x_0}) = \frac{1}{(e^{x_0})'} = \frac{1}{e^{x_0}}$$

$y := e^{x_0} \rightsquigarrow (f^{-1})'(y) = \frac{1}{y}$

$(\ln)'(y) = \frac{1}{y}$

$f(x) = \operatorname{tg} x$ dla $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\qquad f^{-1} = \operatorname{arctg}$

dla dow. $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ zachodzi:

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

$$(\operatorname{tg})'(x) = \operatorname{tg}^2 x + 1 = \frac{1}{\cos^2 x}$$

$$(\operatorname{arctg})'(\operatorname{tg} x) = \frac{1}{\operatorname{tg}^2 x + 1}$$

$y = \operatorname{tg} x$

$$\boxed{(\operatorname{arctg})'(y) = \frac{1}{y^2+1}}$$

Fakt.

$(\ln x)' = \frac{1}{x}$, $x > 0$

$\left.\begin{array}{l}(\operatorname{arctg} x)' = \frac{1}{x^2+1} \\ (\operatorname{arcctg} x)' = \frac{-1}{x^2+1}\end{array}\right\} x \in \mathbb{R}$

$$\underbrace{(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = \frac{-1}{\sqrt{1-x^2}}}_{x \in (-1,1)}$$

Nr.

$$(fg)' = f'g + fg' \qquad (x^n)' = nx^{n-1}$$

$$\left(\sqrt{2x}\arcsin x\right)' = \left(\sqrt{2x}\right)' \arcsin x + \sqrt{2x}\cdot\left(\arcsin x\right)' =$$

$$= \left(\sqrt{2}\cdot\sqrt{x}\right)' \arcsin x + \sqrt{2x}\cdot\frac{1}{\sqrt{1-x^2}} =$$

f. stała

$$= \Big(\underbrace{(\sqrt{2})'}_{=0}\cdot\sqrt{x} + \sqrt{2}\cdot\underbrace{\left(x^{\frac{1}{2}}\right)'}_{=\frac{1}{2}x^{\frac{1}{2}-1}}\Big)\arcsin x + \frac{\sqrt{2x}}{\sqrt{1-x^2}} =$$

$$= \sqrt{2}\cdot\frac{1}{2}x^{-\frac{1}{2}}\arcsin x + \frac{\sqrt{2x}}{\sqrt{1-x^2}}, \qquad x\in(0,1)$$

Tw. Załóżmy, że $f:(a,b)\to(c,d)$, $g:(c,d)\to\mathbb{R}$, $x_0\in(a,b)$ oraz $f'(x_0)$ i $g'(f(x_0))$ istnieją i są skończone. Wówczas

$$\underline{(g\circ f)'(x_0)=g'(f(x_0))\cdot f'(x_0)}.$$

Np.:

$$\underbrace{\left(e^{(x^2)}\right)}_{g(f(x))}' = (g\circ f)'(x) = g'(f(x))\cdot f'(x) = e^{f(x)}\cdot 2x = e^{x^2}\cdot 2x$$

$g(y)=e^y \quad f(x)=x^2$

$g'(y)=e^y \quad f'(x)=2x$

$\underline{g(f(x))=e^{f(x)}=e^{x^2}}$

Np.

$$\left(\sin\left(x^3+1\right)\right)' = g'(\underbrace{x^3+1}_{y}) \cdot f'(x) = \cos(x^3+1)\cdot 3x^2$$

$g(y) = \sin y \qquad f(x) = x^3+1$

$g'(y) = \cos y \qquad f'(x) = 3x^2 + 0$

$$\left(g(f(x))\right)' = (g\circ f)'(x) = g'(f(x))\cdot f'(x)$$

$$\cos(x^3+1)\cdot(x^3+1)' = \cos(x^3+1)\cdot 3x^2$$

Np.

$$\left(\ln^4(x)\right)' = g'(\ln x)\cdot(\ln x)' = 4(\ln x)^3\cdot\frac{1}{x} = \frac{4\ln^3 x}{x}, \quad x>0$$

$g(y) = y^4 \qquad f(x) = \ln x$

$g(f(x)) = g(\ln x) = (\ln x)^4$

$g'(y) = 4y^3$

$(fg)' = f'g + fg'$

Nr.

$$\left(e^{x^2\sin(2x)}\right)' = g'(f(x))\cdot f'(x) = e^{x^2\sin 2x}\cdot\left(x^2\sin(2x)\right)' =$$

$g(y) = e^y \quad f(x) = x^2\sin(2x)$

$g'(y) = e^y$

$$= e^{x^2\sin 2x}\left((x^2)'\cdot\sin 2x + x^2\left(\sin 2x\right)'\right) = e^{x^2\sin 2x}\left(2x\sin 2x + x^2\cdot\cos(2\cancel{x})\,2\right)$$

$$\left\{\begin{array}{l} (\sin(2x))' = \cos(2x)\cdot(2x)' = \cos(2x)\cdot 2 \\ g(y) = \sin y \quad f(x) = 2x \\ g'(y) = \cos y \quad f'(x) = (2)'\cdot x + 2(x)' = 0 + 2\cdot 1 = 2 \end{array}\right.$$

$(\sin x)' = \cos x\cdot(x)' = \cos x$

$g(y) = \sin y \quad f(x) = x$

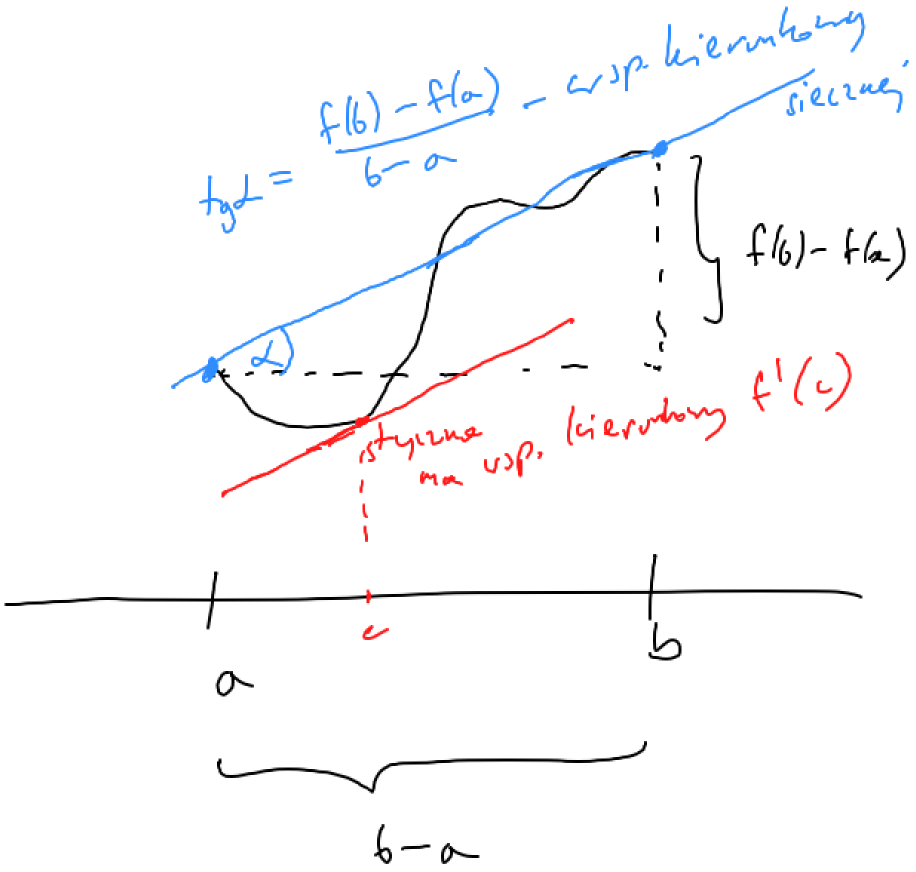
$g'(y) = \cos y$

Tw. (Lagrange'a)

Załóżmy, że $f: [a,b] \to \mathbb{R}$ jest ciągła na $[a,b]$ oraz różniczkowalna na (a,b). Wówczas istnieje punkt $c \in (a,b)$ taki, że

$$\underbrace{f'(c)}_{\text{prędkość chwilowa w chwili } c} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{prędkość średnia od chwili } a \text{ do chwili } b}.$$

$f(t)$ = położenie w t



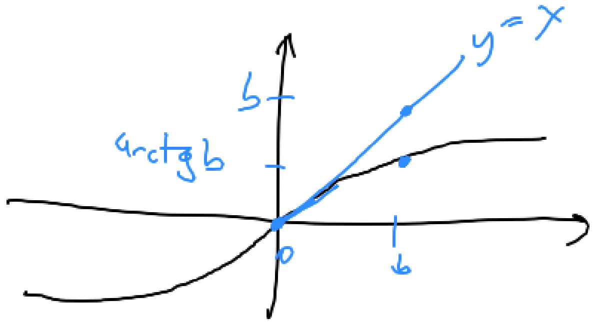
Np. Skorzystamy z tw. Lagrange'a dla $f(x) = \operatorname{arctg} x$ i odcinka $[0, b]$

$f'(x) = \frac{1}{1+x^2}$

Istnieje $c \in (0, b)$ taki, że

$$\frac{1}{1+c^2} = f'(c) = \frac{f(b) - f(0)}{b - 0} = \frac{\operatorname{arctg} b - \operatorname{arctg} 0}{b} = \frac{\operatorname{arctg} b}{b}$$

$$\operatorname{arctg} b = b \cdot \underbrace{\frac{1}{1+c^2}}_{<1} < b \quad \text{dla} \quad b > 0.$$

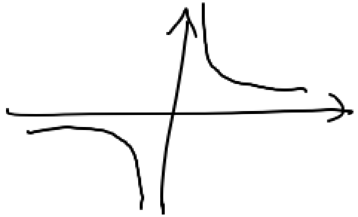


Fakt. Załóżmy, że $f:[a,b]\to\mathbb{R}$ jest ciągła i jest różniczkowalna na (a,b)

- Jeśli $f'>0$ na odcinku (a,b), to f jest rosnąca na $[a,b]$
- -ll- $f'<0$ -ll- ———, to f jest malejąca na $[a,b]$

Uwaga: Można też przyjąć rozważać odcinek otwarty (a,b) lub jednostronnie otwarty $(a,b]$ lub $[a,b)$

Zamiast $[a,b]$



Np: $f(x)=\frac{1}{x}$ $\quad f'(x)=-\frac{1}{x^2}$, $x\neq 0$

$f'<0$ na $(0,\infty)$ $\to$ f jest ~~ros~~ malejąca na $(0,\infty)$

$f'<0$ na $(-\infty,0)$ $\to$ f jest malejąca na $(-\infty,0)$

(Ale f nie jest malejąca na $(-\infty,0)\cup(0,\infty)$!)