Def. $F: I \to \mathbb{R}$ jest funkcją pierwotną funkcji $f: I \to \mathbb{R}$ (gdzie $I \subset \mathbb{R}$ jest odcinkiem), jeśli F jest ciągła na I oraz $F' = f$ wewnątrz I.

Ozn.

$$\int f(x)\,dx = \{F(x) + c : c \in \mathbb{R}\}$$, gdzie F jest pewną funkcją pierwotną f

$$= F(x) + c$$

oznaczenie skrótowe

zamiast

$$\int \sin x\,dx = \{-\cos x + C : C \in \mathbb{R}\}$$

piszemy

$$\int \sin x\,dx = -\cos x + C$$

Fakt. $a, b \in \mathbb{R}$ - stałe, $a \neq 0$ lub $b \neq 0$, $f, g: I \to \mathbb{R}$, F, G - pewne f. pierwotne f, g odpowiednio

$$\int \big(af(x) + bg(x)\big)\,dx = a\int f(x)\,dx + b\int g(x)\,dx = a\{F + c : c \in \mathbb{R}\} + b\{G + d : d \in \mathbb{R}\} =$$

$$= \{a(F+c) : c \in \mathbb{R}\} + \{b(G+d) : d \in \mathbb{R}\} = \{a(F+c) + b(G+d) : c, d \in \mathbb{R}\}$$

$$= \{aF + bG + \underbrace{(ac+bd)} : c, d \in \mathbb{R}\} = \{\underbrace{aF + bG + c'} : c' \in \mathbb{R}\}$$

$$2 \cdot [3,4] = [6,8]$$

Np.

$$\int 2\frac{1}{\sqrt{1-x^2}}\,dx = 2\cdot\int\frac{1}{\sqrt{1-x^2}}\,dx = 2(\arcsin x + c) = 2\arcsin x + 2c$$

$(-1,1)$

$$\| \quad 2\arcsin x + \tilde{c}$$

Fakt. (całkowanie przez podstawienie)

Załóżmy, że $f: I \to \mathbb{R}$, $\varphi: I \to \mathbb{R}$. Niech f ma funkcję pierwotną F, niech φ' istnieje i będzie ciągła wewnątrz I, przy czym I jest odcinkiem otwartym. Wtedy

$$\int f(\varphi(x))\,\varphi'(x)\,dx = F(\varphi(x)) + C$$

Spr. $F\circ\varphi$ jest f. ciągła, $(F\circ\varphi)'(x) = F'(\varphi(x))\cdot\varphi'(x) = f(\varphi(x))\,\varphi'(x)$

Np.

$$\int \cos(x^3)\cdot 3x^2\,dx = \sin(x^3) + C \qquad \left(f(x) = \cos x,\ \varphi(x) = x^3,\ F(x) = \sin x \right)$$

Zapis:

$$\int \cos(x^3)\cdot 3x^2\,dx = \left\{\begin{aligned} & t = x^3 \quad \Big|\ \frac{dt}{dx} = 3x^2 \\ & 1\,dt = 3x^2\,dx \end{aligned}\right\} = \int \cos(t)\,dt = \sin t + C = \sin(x^3) + C$$

Npr.

$$\int e^{-x^2}\,x\,dx = \left\{\begin{aligned} & t = -x^2 \\ & dt = -2x\,dx \quad \Big|:(-2) \\ & -\tfrac{1}{2}\,dt = x\,dx \end{aligned}\right\} = \int e^{t}\cdot\left(-\tfrac{1}{2}\right)dt = -\frac{1}{2}\int e^{t}\,dt =$$

$$= -\frac{1}{2}e^{t} + C = -\frac{1}{2}e^{-x^2} + C$$

$$\parallel$$

$$-\frac{1}{2}\int e^{-x^2}(-2x)\,dx$$

$$\int \sin(3x+1)\,dx = \left\{ \begin{aligned} t &= 3x+1 \\ dt &= 3\,dx \\ \tfrac{1}{3}dt &= dx \end{aligned} \right\} = \int \sin t \cdot \frac{1}{3}\,dt = \frac{1}{3}\int \sin t\,dt = \frac{1}{3}(-\cos t) + C =$$

$$= \frac{1}{3}\left(-\cos(3x+1)\right) + C$$

$$\parallel$$

$$-\frac{1}{3}\cos(3x+1) + C$$

$$\left(-\frac{1}{3}\cos(3x+1)\right)' = \frac{1}{3}\sin(3x+1)\cdot(3x+1)' = \frac{1}{3}\,3\sin(3x+1)$$

Całkowanie przez części

$f, g: (a,b) \to \mathbb{R}$, f', g' istnieją i są ciągłe na (a,b)

Wtedy

$$\int f'(x) g(x)\, dx = f(x) g(x) - \int f(x) g'(x)\, dx$$

Spr. pochodna lewej strony $(x) = f'(x) g(x)$

-||- prawej strony $(x) = \left(f'(x) g(x) + f(x) g'(x)\right) - f(x) g'(x) = f'(x) g(x)$

Np

$$\int x \cdot \sin x\, dx = \int \left(\frac{x^2}{2}\right)' \sin x\, dx = \frac{x^2}{2} \cdot \sin x - \int \frac{x^2}{2} \cdot (\sin x)'\, dx = \frac{x^2}{2} \sin x - \int \frac{x^2}{2} \cdot \cos x\, dx$$

↑ to nie jest dobra droga

$$\| $$

$$\int \sin x \cdot x\, dx = \int (-\cos x)' x\, dx = -\cos x \cdot x - \int (-\cos x) \cdot \underbrace{(x)'}_{= 1} dx =$$

$$= -x \cos x + \int \cos x\, dx = -x \cos x + \sin x + C$$

Nr.

$$\int x e^{-2x} dx = \left| \begin{array}{l} t = -2x \\ dt = -2\,dx \\ -\frac{1}{2}dt = dx \end{array} \right| = -\frac{1}{2}\int -\frac{t}{2} \cdot e^{t}\, dt = \frac{1}{4}\int t e^{t}\, dt =$$

$$x = -\frac{t}{2}$$

$$\int f g' = fg - \int f' g$$

$$= \frac{1}{4}\int t\,(e^{t})'\, dt = \frac{1}{4}\left[t e^{t} - \int (t)' \cdot e^{t}\, dt \right] = \frac{1}{4} t e^{t} - \frac{1}{4}\int e^{t} dt =$$

$$= \frac{1}{4} t e^{t} - \frac{1}{4} e^{t} + C = \frac{1}{4}(-2x)\, e^{-2x} - \frac{1}{4} e^{-2x} + C = \underline{\underline{\frac{-x}{2} e^{-2x} - \frac{1}{4} e^{-2x} + C}}$$

Sprawdzimy:

$$\left(-\frac{x}{2} \cdot e^{-2x} - \frac{1}{4} e^{-2x}\right)' = \left(-\frac{x}{2}\right)' e^{-2x} + \left(-\frac{x}{2}\right)\left(e^{-2x}\right)' - \frac{1}{4}\left(e^{-2x}\right)' =$$

$$= -\frac{1}{2} e^{-2x} - \frac{x}{2} e^{-2x} \cdot (-2) - \frac{1}{4} e^{-2x}(-2) = \underbrace{-\frac{1}{2} e^{-2x}} + x e^{-2x} + \underbrace{\frac{1}{2} e^{-2x}} = x e^{-2x}$$

Wpr.

$$\left(\int f'g = fg - \int fg' \right.$$

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx = \int (x)' \ln x \, dx = x \ln x - \int x (\ln x)' dx =$$

$$= x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int 1 dx = x \ln x - x + C$$

$$\left| \begin{array}{ll} f' = 1 & g = \ln x \\ f = x & g' = \frac{1}{x} \end{array} \right|$$

$$\left\{ \begin{array}{l} (t^{n+1})' = (n+1) t^n \\ \left(\frac{1}{n+1} t^{n+1}\right)' = t^n \\ \int t^n dt = \frac{1}{n+1} t^{n+1} + C, \quad n \neq -1 \end{array} \right.$$

Wpr.

$$\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx = \int (x)' \arcsin x \, dx = x \arcsin x - \int x (\arcsin x)' dx =$$

$$= x \arcsin x - \frac{1}{2} \int 2x \cdot \frac{1}{\sqrt{1-x^2}} dx = \left\{ \begin{array}{l} t = 1 - x^2 \\ dt = -2x \, dx \end{array} \right\} = x \arcsin x + \frac{1}{2} \int \frac{1}{\sqrt{t}} dt =$$

$$= x \arcsin x + \frac{1}{2} \int t^{-\frac{1}{2}} dt = x \arcsin x + \cancel{\frac{1}{2}} \cdot \cancel{\frac{1}{-\frac{1}{2}+1}} t^{-\frac{1}{2}+1} + C = x \arcsin x + \underbrace{(1-x^2)^{\frac{1}{2}}}_{\sqrt{1-x^2}} + C$$

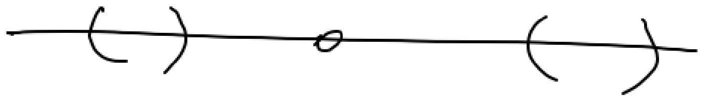
dla $x>0$:

$$\int \frac{1}{x}\,dx = \ln x + C = \ln|x| + C$$

dla $x<0$:

$$\int \frac{1}{x}\,dx = \left\{ \begin{array}{l} x=-t \\ dx=-dt \\ t>0 \end{array} \right\} = \int \frac{1}{-t}\cdot(-1)\,dt = \int \frac{1}{t}\,dt = \ln t + C = \ln(-x) + C = \ln|x| + C$$

$$\Rightarrow \boxed{\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C, & n \neq -1 \\ \ln|x| + C, & n = -1 \end{cases}}$$



$a \in \mathbb{R}$

$$\int \frac{1}{x-a}\,dx = \left| \begin{array}{l} t = x-a \\ dt = dx \end{array} \right| = \int \frac{1}{t}\,dt = \ln|t| + c = \ln|x-a| + c$$

$k>1$:

$$\int \frac{1}{(x-a)^k}\,dx = \left| \quad -||- \quad \right| = \int \frac{1}{t^k}\,dt = \int t^{-k}\,dt = \frac{t^{-k+1}}{-k+1} + C = \frac{1}{-k+1}\,\frac{1}{(x-a)^{k-1}} + C$$

$\int \frac{P(x)}{Q(x)}\,dx$, P, Q — wielomiany

z algebry wiadomo, że $\frac{P}{Q} = \underbrace{I}_{\text{wielomian}} + $ (suma ułamków prostych)

ułamki proste to funkcje wymierne postaci:

- $\frac{C}{(x-a)^k}$ I rodzaju
- $\frac{ax+b}{(x^2+px+q)^k}$ II rodzaju, $p^2-4q<0$

Przypomnienie:

np.

$$\frac{x^2+7}{(\underbrace{x^2+x+1}_{\Delta=-3<0})^2(x-1)^3x} = I(x) + \boxed{\frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}} + \frac{E}{x-1} + \frac{F}{(x-1)^2} + \frac{G}{(x-1)^3} + \frac{H}{x}$$

??

wielomian, tutaj zerowy

to już umiemy całkować!

gdzie $A, B, \dots, H$ — pewne stałe

Np.

- $$\int \frac{2x+3}{\underbrace{x^2+3x+4}_{\Delta<0}}\,dx = \left\{ \begin{array}{l} t = x^2+3x+4 \\ dt = (2x+3)\,dx \end{array} \right\} = \int \frac{dt}{t} = \ln|t| + C =$$

$$= \ln|x^2+3x+4| + C = \ln(x^2+3x+4) + C$$

$\uparrow$ przyjmuje tylko wartości dodatnie

- $$\int \frac{2x+7}{x^2+3x+4}\,dx = \int \frac{(2x+3)+4}{x^2+3x+4}\,dx = \underbrace{\int \frac{2x+3}{x^2+3x+4}\,dx}_{\text{policzyliśmy wyżej}} + \underbrace{4\int \frac{1}{x^2+3x+4}\,dx}_{?}$$

$(x^2+3x+4)' = 2x+3$

- $$\int \frac{3x+8}{x^2+3x+4}\,dx = \int \frac{\frac{3}{2}(2x+3)+\frac{7}{2}}{x^2+3x+4}\,dx = \underbrace{\frac{3}{2}\int \frac{2x+3}{x^2+3x+4}\,dx}_{\text{wiemy}} + \frac{7}{2}\underbrace{\int \frac{dx}{x^2+3x+4}}_{?}$$

$$\int \frac{1}{\underbrace{x^2+3x+4}} \, dx = \int \frac{1}{\underbrace{\left(x+\frac{3}{2}\right)^2}_{x^2+3x+\frac{9}{4}} + \frac{7}{4}} \, dx =$$

$$\left| \begin{array}{l} t = \frac{2}{\sqrt{7}}\left(x+\frac{3}{2}\right) \\ dt = \frac{2}{\sqrt{7}} \, dx \\ dx = \frac{\sqrt{7}}{2} \, dt \end{array} \right|$$

$$= \frac{1}{\frac{7}{4}} \int \frac{dx}{\frac{4}{7}\left(x+\frac{3}{2}\right)^2 + 1} = \frac{4}{7} \int \frac{dx}{\left(\frac{2}{\sqrt{7}}\left(x+\frac{3}{2}\right)\right)^2 + 1} =$$

$$\boxed{\int \frac{dt}{t^2+1} = \operatorname{arctg} t + C}$$

$$= \frac{4}{7} \int \frac{\frac{\sqrt{7}}{2} \, dt}{t^2+1} = \frac{2\sqrt{7}}{7} \int \frac{dt}{t^2+1} =$$

$$= \frac{2\sqrt{7}}{7} \operatorname{arctg} t + C = \underbrace{\frac{2\sqrt{7}}{7} \operatorname{arctg}\left(\frac{2}{\sqrt{7}}\left(x+\frac{3}{2}\right)\right) + C}$$

$$\int \frac{1}{\underbrace{2x^2+4x+10}_{\Delta=16-80<0}}\,dx = \frac{1}{2}\int \frac{dx}{x^2+2x+5} = \frac{1}{2}\int \frac{dx}{(x+1)^2+4} =$$

$$= \frac{1}{2}\,\frac{1}{4}\int \frac{dx}{\frac{1}{4}(x+1)^2+1} = \frac{1}{8}\int \frac{dx}{\left(\frac{x+1}{2}\right)^2+1} = \left\{ \begin{array}{l} t = \frac{x+1}{2} \\ dt = \frac{1}{2}\,dx \\ 2dt = dx \end{array} \right\} =$$

$$\left\{ \; 2x^2+4x+10 = 8\left(\left(\frac{x+1}{2}\right)^2+1\right) \right.$$

$$= \frac{1}{8}\int \frac{2\,dt}{t^2+1} = \frac{1}{4}\operatorname{arctg} t + C = \underline{\frac{1}{4}\operatorname{arctg}\left(\frac{x+1}{2}\right)+C}$$