$$\int \underbrace{\frac{x^4+4x^3+11x^2+6x+2}{x^3+3x^2+3x-7}}_{= f(x)}\,dx$$

$$\begin{array}{r} 17 \\ \hline 120:7 \\ -7 \\ \hline 50 \\ -49 \\ \hline 1 \end{array}$$

$120:7 = 17$ reszty 1

$$\frac{120}{7} = 17 + \frac{1}{7}$$

I jeśli funkcja wymierna jest niewłaściwa (tzn. stopień licznika $\geq$ stopień mianownika), to wykonujemy dzielenie wielomianów:

$$\begin{array}{l} \quad x+1 \\ \hline (x^4+4x^3+11x^2+6x+2):(x^3+3x^2+3x-7) \\ -(x^4+3x^3+3x^2-7x) \\ \hline \quad x^3+8x^2+13x+2 \\ \quad -(x^3+3x^2+3x-7) \\ \hline \quad\quad 5x^2+10x+9 \end{array}$$

$$\Rightarrow f(x) = x+1+\frac{5x^2+10x+9}{x^3+3x^2+3x-7}$$

II funkcję wymierną właściwą

$$\frac{5x^2+10x+9}{x^3+3x^2+3x-7} = \frac{5x^2+10x+9}{(x-1)(x^2+4x+7)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4x+7}$$

I rodzaju $\dfrac{A}{(x-a)^k}$ II rodzaju $\dfrac{Bx+C}{(x^2+px+q)^k}$

$p^2-4q<0$

rozkładamy na sumę ułamków prostych

. zaczynamy od rozkładu mianownika na czynniki nierozkładalne

wsp. są całkowite, spróbujmy odgadnąć pierwiastek całkowity:

dzielniki wyrazu wolnego $\{-7,-1,1,7\}$

sp. że 1 jest pierwiastkiem, więc mianownik dzieli się przez $(x-1)$

wykonujemy dzielenie (można też użyć schematu Hornera lub grupowania wyrazów)

$$\begin{array}{l} \quad\ \ x^2+4x+7 \\ \hline (x^3+3x^2+3x-7):(x-1) \\ -(x^3-x^2) \\ \hline \qquad 4x^2+3x-7 \\ \quad\ -(4x^2-4x) \\ \hline \qquad\qquad 7x-7 \\ \qquad\ -(7x-7) \\ \hline \qquad\qquad 0 \end{array}$$

reszta powinna wyjść zero

$$x^3+3x^2+3x-7 = (x-1)\underbrace{(x^2+4x+7)}$$

$\Delta = 16-28<0$

to jest trójmian nierozkładalny (nad $\mathbb{R}$)

$$\frac{5x^2+10x+9}{x^3+3x^2+3x-7} = \frac{\overset{2}{A}}{x-1} + \frac{\overset{3}{B}x+\overset{5}{C}}{x^2+4x+7} = \frac{A(x^2+4x+7)+(Bx+C)(x-1)}{(x-1)(x^2+4x+7)}$$

równe

Znajdujemy współczynniki:

$$5x^2+10x+9 = A(x^2+4x+7)+(Bx+C)(x-1) = (A+B)x^2+x(4A-B+C)+(7A-C)$$

Tworzymy układ 3 równań na wsp. A, B, C, są 2 sposoby:

- porównujemy współczynniki

$$\begin{cases} A+B=5 \\ 4A-B+C=10 \\ 7A-C=9 \end{cases}$$

- wstawiamy jakieś 3 wartości za x:

$$\begin{aligned} x=1:&\quad 24=12A \rightarrow A=2 \\ x=0:&\quad 9=7A-C \rightarrow 9=14-C \rightarrow C=5 \\ x=-1:&\quad 4=4A+(C-B)(-2)=4A+2B-2C \end{aligned}$$

$$\rightarrow 4=8+2B-10$$

$$6=2B$$

$$B=3$$

$A=2$

$$\Rightarrow \quad f(x) = x + 1 + \frac{2}{x-1} + \frac{3x+5}{x^2+4x+7}$$

III wstawiamy i obliczamy całki z wielomianu i ułamków prostych I rodzaju:

$$\int f(x)\,dx = \int x\,dx + \int 1\,dx + \int \frac{2}{x-1}\,dx + \int \frac{3x+5}{x^2+4x+7}\,dx$$

$$= \frac{x^2}{2} + x + 2\ln|x-1| + \underbrace{\int \frac{3x+5}{x^2+4x+7}\,dx}_{\text{to zostaje}}$$

$$\int x^n\,dx = \begin{cases} \frac{x^{n+1}}{n+1} + C, & n \neq -1 \\ \ln|x| + C, & n = -1 \end{cases}$$

$$\int \frac{1}{(x-a)}\,dx = \ln|x-a| + C$$

$$\int \frac{1}{(x-a)^k}\,dx = \quad \boxed{k>1}$$

$$= \int (x-a)^{-k}\,dx$$

$$= \frac{(x-a)^{-k+1}}{-k+1} + C$$

IV obliczamy całki z ułamków prostych II rodzaju

$$\int \frac{3x+5}{x^2+4x+7}\,dx = \int \frac{\frac{3}{2}(2x+4)-1}{x^2+4x+7}\,dx = \frac{3}{2}\int \frac{2x+4}{x^2+4x+7}\,dx - \int \frac{dx}{x^2+4x+7}$$

$$(x^2+4x+7)' = 2x+4$$

- $$\int \frac{2x+4}{x^2+4x+7}\,dx = \left| \begin{matrix} t = x^2+4x+7 \\ dt = (2x+4)\,dx \end{matrix} \right| = \int \frac{dt}{t} = \ln|t| + C = \ln|x^2+4x+7| + C$$

- $$\int \frac{dx}{x^2+4x+7} = \int \frac{dx}{(x+2)^2+3} = \frac{1}{3}\int \frac{dx}{\frac{(x+2)^2}{3}+1} = \frac{1}{3}\int \frac{dx}{\left(\frac{x+2}{\sqrt{3}}\right)^2+1} = \left| \begin{matrix} u = \frac{x+2}{\sqrt{3}} \\ du = \frac{1}{\sqrt{3}}\,dx \\ \sqrt{3}\,du = dx \end{matrix} \right|$$

$$= \frac{1}{3}\int \frac{\sqrt{3}\,du}{u^2+1} = \frac{\sqrt{3}}{3}\,\mathrm{arctg}(u) + C = \frac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{x+2}{\sqrt{3}}\right) + C$$

$$\left(\frac{x+2}{\sqrt{3}}\right)'$$

$$\left(\sqrt{3}\,\mathrm{arctg}\,\frac{x+2}{\sqrt{3}}\right)' = \frac{1}{\left(\frac{x+2}{\sqrt{3}}\right)^2+1}\cdot\frac{1}{\sqrt{3}}\,\sqrt{3}$$

Odp.

$$\int f(x)\,dx = \frac{x^2}{2} + x + 2\ln|x-1| + \frac{3}{2}\ln(x^2+4x+7) - \frac{\sqrt{3}}{3}\operatorname{arctg}\left(\frac{x+2}{\sqrt{3}}\right) + C$$

Całkowanie funkcji trygonometrycznych

$\int R(\cos x, \sin x)\,dx$, R – f. wymierna dwóch zmiennych

np. $\int \frac{\cos x}{\sin^2 x + \cos x + 4}\,dx$ $\qquad \int \sin^2 x \cos x\,dx$ $\qquad \int \frac{dx}{3-\sin x}$ $\qquad \int \operatorname{tg} x\,dx$, $\operatorname{tg} x = \frac{\sin x}{\cos x}$

Podstawienia:

- $t = \sin x$ lub $t = \cos x$

np. $\int \sin x \cos^4 x \, dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x \, dx \\ -dt = \sin x \, dx \end{array} \right| = -\int t^4 \, dt = -\frac{t^5}{5} + C = \boxed{-\frac{\cos^5 x}{5} + C}$

$\cos^3 x \cdot \cos x$

$= \left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \\ \cos^3 x = ?? \end{array} \right| = \int t(1-t^2) \cdots dt$

?? $\pm\sqrt{1-t^2}$?? $\leftarrow$ to nie jest dobry sposób

$\cos^2 x = 1 - \sin^2 x = 1 - t^2$

$\cos x = \pm\sqrt{1-t^2}$??

$\int \sin x \cos^3 x \, dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x \\ -dt = \sin x \, dx \end{array} \right| = -\int t^3 dt = \frac{-t^4}{4} + C = \frac{-\cos^4 x}{4} + C$

$\cos^2 x \underbrace{\cos x \, dx}$

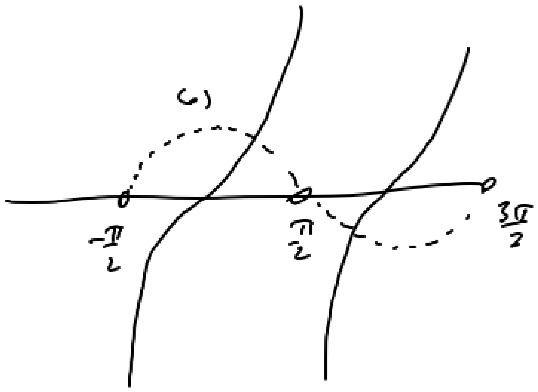
$= \left| \begin{array}{ll} t = \sin x & \cos^2 x = 1 - t^2 \\ dt = \cos x \, dx & \end{array} \right| = \int t(1-t^2) \, dt = \int (t - t^3) dt = \frac{t^2}{2} - \frac{t^4}{4} + \tilde{C} = \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + \tilde{C}$

$\frac{-\cos^4 x}{4} + C = \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4}$

$\Rightarrow$ $\frac{1}{4}$ $(x=0)$

$$\int tg\,x\,dx = \int \frac{\sin x}{\cos x}\,dx = \left| \begin{array}{l} t=\cos x \\ dt = -\sin x\,dx \\ -dt = \sin x\,dx \end{array} \right| = \int \frac{-dt}{t} = -\ln|t|+C$$

$$= -\ln|\cos x| + C$$



$\|$

$t = \sin x$

$dt = \cos x\,dx$

$$\int \frac{\sin x\,\boxed{\cos x\,dx}}{\cos^2 x} = \int \frac{t}{1-t^2}\,dt = \int \left(\frac{A}{t-1} + \frac{B}{t+1}\right) dt = \dots$$

• podstawienie $t = \mathrm{tg}\, x$ ma zastosowanie, gdy funkcja podcałkowa jest funkcją $\mathrm{tg}\, x$, $\mathrm{ctg}\, x$, $\cos^2 x$, $\sin^2 x$

$$\int \frac{1}{\mathrm{tg}^2 x + 4}\, dx = \left| \begin{array}{l} t = \mathrm{tg}\, x \\ dt = (\mathrm{tg}^2 x + 1)\, dx \\ \dfrac{dt}{t^2+1} = dx \end{array} \right| = \int \frac{1}{t^2+4} \, \frac{dt}{t^2+1}$$

$$\frac{1}{(t^2+4)(t^2+1)} = \frac{At+B}{t^2+4} + \frac{Ct+D}{t^2+1} = \frac{(At+B)(t^2+1) + (Ct+D)(t^2+4)}{(t^2+4)(t^2+1)} \ldots$$

wyznaczamy wsp. A, B, C, D

...

$$\int \operatorname{tg}x \, dx = \left| \begin{array}{l} t = \operatorname{tg}x \\ dt = (\operatorname{tg}^2 x + 1)\, dx \\ \frac{dt}{t^2+1} = dx \end{array} \right| = \int t \, \frac{dt}{t^2+1} = \frac{1}{2} \int \frac{2t}{t^2+1}\, dt =$$

$$= \frac{1}{2} \ln |t^2+1| + C = \underline{\frac{1}{2} \ln (\operatorname{tg}^2 x + 1) + C}$$

$\ln(a^b) = b \ln a,$
$a > 0$

$$\operatorname{tg}^2 x + 1 = \frac{\sin^2 x}{\cos^2 x} + 1 = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\Rightarrow \ln(\operatorname{tg}^2 x + 1) = \ln\left((\cos x)^{-2}\right) = \ln\left(|\cos x|^{-2}\right) = -2 \ln|\cos x|$$

$$\int \frac{1}{\sin^2 x + 2\cos^2 x + 3}\,dx = \left| \begin{array}{l} t = \operatorname{tg} x \\ \frac{dt}{t^2+1} = dx \end{array} \right| = \int \frac{1}{\frac{t^2}{t^2+1} + 2\frac{1}{t^2+1} + 3}\,\frac{dt}{t^2+1} =$$

$$\operatorname{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x} \to \cos^2 x \operatorname{tg}^2 x = 1 - \cos^2 x$$

$$\cos^2 x(\operatorname{tg}^2 x + 1) = 1$$

$$\boxed{\cos^2 x = \frac{1}{\operatorname{tg}^2 x + 1}}$$

$$\operatorname{tg}^2 x = \frac{\sin^2 x}{1-\sin^2 x}$$

$$\operatorname{tg}^2 x(1-\sin^2 x) = \sin^2 x$$

$$\operatorname{tg}^2 x = \sin^2 x(1+\operatorname{tg}^2 x)$$

$$\boxed{\sin^2 x = \frac{\operatorname{tg}^2 x}{\operatorname{tg}^2 x + 1}}$$

$$= \int \frac{dt}{t^2 + 2 + 3(t^2+1)} = \int \frac{dt}{4t^2+5} =$$

$$= \frac{1}{5}\int \frac{dt}{\frac{4}{5}t^2+1} = \frac{1}{5}\int \frac{dt}{\left(\frac{2t}{\sqrt{5}}\right)^2+1} = \frac{1}{5}\operatorname{arctg}\frac{2t}{\sqrt{5}} \cdot \frac{\sqrt{5}}{2} + C$$

$$= \frac{\sqrt{5}}{10}\operatorname{arctg}\frac{2\operatorname{tg} x}{\sqrt{5}} + C$$

$$\int \sin^4 x \cos^2 x \, dx = \left| \begin{array}{l} t = \operatorname{tg} x \\ \dfrac{dt}{t^2+1} = dx \\ \sin^2 x = \dfrac{\operatorname{tg}^2 x}{\operatorname{tg}^2 x + 1} = \dfrac{t^2}{t^2+1} \\ \cos^2 x = \dfrac{1}{\operatorname{tg}^2 x + 1} = \dfrac{1}{t^2+1} \end{array} \right| = \int \left(\frac{t^2}{t^2+1}\right)^2 \frac{1}{t^2+1} \, \frac{dt}{t^2+1} =$$

$$= \int \frac{t^4}{(t^2+1)^4} \, dt$$

bardzo kłopotliwe rachunkowo

• <u>podstawienie uniwersalne</u>

$$t = \operatorname{tg}\frac{x}{2}$$

$$dt = \left(\operatorname{tg}^2\frac{x}{2}+1\right)\cdot\frac{1}{2}\,dx$$

$$\boxed{\frac{2\,dt}{t^2+1} = dx}$$

np.

$$\int\frac{\boxed{dx}}{2-\cos x} = \int\frac{1}{2-\frac{1-t^2}{1+t^2}}\ \boxed{\frac{2\,dt}{t^2+1}} =$$

$$= \int\frac{2\,dt}{2t^2+2-(1-t^2)} = \int\frac{2\,dt}{3t^2+1} = 2\int\frac{dt}{(\sqrt{3}t)^2+1} =$$

$$= \frac{2}{\sqrt{3}}\operatorname{arctg}(\sqrt{3}t)+C = \boxed{\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\sqrt{3}\operatorname{tg}\frac{x}{2}\right)+C}$$

$$\left[\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2} = \frac{\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}{\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)}\ \frac{:\cos^2\frac{x}{2}}{:\cos^2\frac{x}{2}} = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}\right]$$

$$\left[\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}\ \frac{:\cos^2\frac{x}{2}}{:\cos^2\frac{x}{2}} = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2t}{1+t^2}\right]$$