35 f

$$\lim_{x\to 0}\frac{e^{2x}-1}{\sin 3x} = \lim_{x\to 0}\frac{\underbrace{\frac{e^{2x}-1}{2x}}_{\to 1}\,2\cancel{x}}{\underbrace{\frac{\sin 3x}{3x}}_{\to 1}\,3\cancel{x}} = \frac{2}{3}$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=1$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$$

g)

$$\lim_{x\to 0}\frac{\ln(1+\sqrt[3]{x})}{x} = \lim_{x\to 0}\frac{\underbrace{\frac{\ln(1+\sqrt[3]{x})}{\sqrt[3]{x}}}_{\to 1}}{\underbrace{\sqrt[3]{x^2}}_{\to 0^+}} = \infty$$

$a>0,\ a\neq 1$

$$\lim_{x\to 0}\frac{a^x-1}{x} = \lim_{x\to 0}\underbrace{\frac{e^{x\ln a}-1}{\underbrace{x\ln a}_{\to 0}}}_{\to 1}\cdot\ln a$$

$$=\ln a$$

h)

$$\lim_{x\to -2}\frac{\ln(x^2-3)}{x+2} = \lim_{x\to -2}\underbrace{\frac{\ln(1+(x^2-4))}{(x+2)(x-2)}}_{\downarrow\, 1}\cdot\underbrace{(x-2)}_{\downarrow\, -4} = -4$$

$$\left\|\; \begin{aligned} &y = x+2 \to 0\\ &x = y-2 \end{aligned}\right.$$

$$\lim_{y\to 0}\frac{\ln((y-2)^2-3)}{y} = \lim_{y\to 0}\frac{\ln(y^2-4y+1)}{y} = \lim_{y\to 0}\underbrace{\frac{\ln((y^2-4y)+1)}{y^2-4y}}_{\downarrow\, 1}\underbrace{(y-4)}_{\downarrow\, -4}$$

i) $$\lim_{x\to 1}\frac{x^{\pi}-x^{e}}{x-1}=\lim_{x\to 1}\frac{e^{\pi\ln x}-e^{e\ln x}}{x-1}=\lim_{x\to 1}\frac{\left(e^{\pi\ln x}-1\right)-\left(e^{e\ln x}-1\right)}{x-1}=$$

$$=\lim_{x\to 1}\left(\frac{e^{\pi\ln x}-1}{\underbrace{\pi\ln x}_{\downarrow\, 1}}\cdot\underbrace{\frac{\pi\ln(\overbrace{x-1}^{\to 0}+1)}{\underbrace{x-1}_{\downarrow\, 0}}}_{\downarrow\, 1}-\frac{e^{e\ln x}-1}{\underbrace{e\ln x}_{\downarrow\, 1}}\cdot\underbrace{\frac{e\ln x}{x-1}}_{\downarrow\, 1}\right)=1\cdot\pi\cdot 1-1\cdot e\cdot 1=\pi-e$$

j) $\lim\limits_{x\to 0}(1+2x)^{\frac{1}{x}} = \lim\limits_{\cancel{x}\, t\to 0}(1+t)^{\frac{2}{t}} = \lim\limits_{t\to 0}\left((1+t)^{\frac{1}{t}}\right)^2 = e^2$ $\qquad 2x = t$

$a = e^{\ln a} \quad (a>0)$

$\left(1+\frac{1}{a_n}\right)^{a_n} \to e$ jeśli $a_n \to \infty$ lub $a_n \to -\infty$

$|a_n| \to \infty$

$1+2x = e^{\ln(1+2x)}$

$(1+2x)^{\frac{1}{x}} = \left(e^{\ln(1+2x)}\right)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+2x)} \xrightarrow[x\to 0]{} e^2$

wystarczy policzyć $\lim\limits_{x\to 0}\frac{1}{x}\ln(1+2x) = \lim \frac{\ln(1+2x)}{2x}\cdot 2 = 2$

$\begin{cases} x^3,\ \sqrt{x} & \text{– f. potęg.} \\ 2^x,\ e^x & \text{– f. wykładn.} \end{cases}$

k)

$$\lim_{x\to 0}\left(1+\operatorname{tg}(2x)\right)^{\operatorname{ctg}x} = \lim_{x\to 0} e^{\operatorname{ctg}x\cdot\ln(1+\operatorname{tg}(2x))} = e^2$$

$$\lim_{x\to 0}\operatorname{ctg}x\cdot\ln\left(1+\operatorname{tg}(2x)\right) = \lim_{x\to 0}\frac{\ln(1+\operatorname{tg}(2x))}{\operatorname{tg}x} =$$

$$= \lim_{x\to 0}\frac{\ln(1+\operatorname{tg}(2x))}{\operatorname{tg}(2x)}\cdot\frac{\operatorname{tg}(2x)}{\operatorname{tg}(x)} = \lim_{x\to 0}\underbrace{\frac{\ln(1+\operatorname{tg}(2x))}{\operatorname{tg}2x}}_{1}\cdot\frac{\sin 2x\cdot 2\cancel{x}}{\underbrace{\cos 2x}_{1}\cdot 2x}\cdot\frac{\overset{1}{\cos x}\cdot x}{\sin x\cdot\cancel{x}} = 2$$

$$\lim_{x\to 0}\frac{\operatorname{tg}x}{x} = 1$$

l)

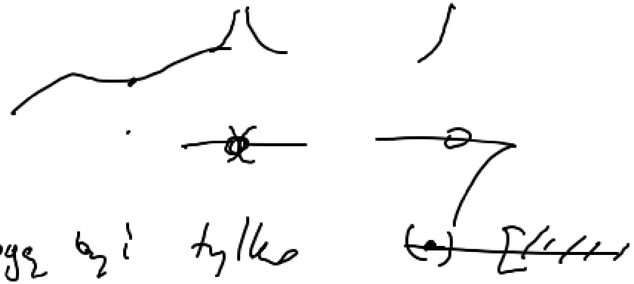
$$\lim_{x\to 0}\frac{\sqrt[3]{1+x}-\sqrt[6]{1-x}}{x} = \lim_{x\to 0}\frac{\left(\sqrt[3]{1+x}-1\right)-\left(\sqrt[6]{1-x}-1\right)}{x} = \frac{1}{3}-\frac{1}{6} = \frac{1}{6}$$

$$\lim_{x\to 0}\frac{\sqrt[3]{1+x}-1}{x} = \lim_{x\to 0}\frac{(1+x)^{\frac{1}{3}}-1}{x} = \lim_{x\to 0}\underbrace{\frac{e^{\frac{1}{3}\ln(1+x)}-1}{\frac{1}{3}\ln(1+x)}}_{1}\cdot\frac{1}{3}\underbrace{\frac{\ln(1+x)}{x}}_{1} = \frac{1}{3}$$

$$\lim_{x\to 0}\frac{\sqrt[6]{1-x}-1}{x} = \lim_{x\to 0}\frac{(1-x)^{\frac{1}{6}}-1}{x} = \lim_{x\to 0}\underbrace{\frac{e^{\frac{1}{6}\ln(1-x)}-1}{\frac{1}{6}\ln(1-x)}}_{1}\cdot\frac{-\frac{1}{6}\ln(1-x)}{-x} = \frac{1}{6}$$

36

$$f(x) = \frac{x^3+x^2}{x^2-4}$$



f jest ciągła, jako elementarna, więc asymptoty pionowe mogą być tylko na „krańcach" dziedziny

$(-\infty,3)\cup(3,7)\cup(7,\infty)$

$0 \neq x^2-4$

$x\neq 2 \quad x\neq -2$ — dziedzina

Trzeba liczyć granice $\lim_{x\to 2^-} f(x)$, $\lim_{x\to 2^+} f(x)$, $\lim_{x\to -2^-} f(x)$, $\lim_{x\to -2^+} f(x)$

$$\lim_{x\to 2^+}\frac{x^3+x^2}{x^2-4} = \lim_{x\to 2^+}\frac{x^3+x^2}{(x-2)(x+2)} = \frac{8+4}{0^+\cdot 4} = +\infty$$

$$\lim_{x\to 2^-}\frac{x^3+x^2}{(x-} = \frac{8+4}{0^-\cdot 4} = -\infty$$

$$x\to -2^- \quad \frac{-8+4}{-4\cdot 0^-} = -\infty$$

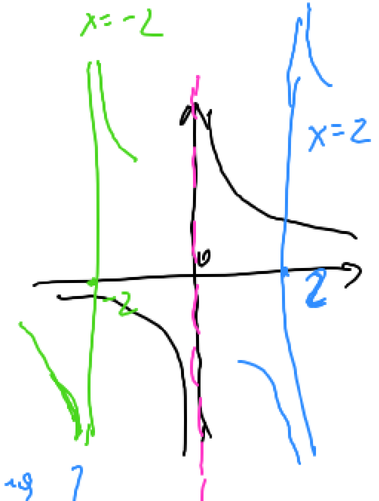
$$x\to -2^+ \quad = \frac{-8+4}{-4\cdot 0^+} = +\infty$$

x=2 jest asymptotą pionową prawostronną

x=2 — " — lewostronną

$\Leftrightarrow$ x=2 jest as. pionową (dwustronną)

x=-2 jest asymptotą pionową



$$f(x)=\frac{x^3+x^2}{x^2-4}$$

$$a_{\pm}=\lim_{x\to\pm\infty}\frac{\cancel{x^3}+x^{\cancel{2}}}{\cancel{x^2}-4}\cdot\frac{1}{\cancel{x}}=\lim_{x\to\pm\infty}\frac{\cancel{x^2}\left(1+\frac{1}{x}\right)}{\cancel{x^2}\left(1-\frac{4}{x^2}\right)}=1$$

$$b_{\pm}=\lim_{x\to\pm\infty}\left(\frac{x^3+x^2}{x^2-4}-x\right)=\lim_{x\to\pm\infty}\frac{\cancel{x^3}+x^2-x(\cancel{x^2}-4)}{x^2-4}=$$

$$=\lim_{x\to\pm\infty}\frac{\cancel{x^2}\left(1+\frac{4}{x}\right)}{\cancel{x^2}\left(1-\frac{4}{x^2}\right)}=1$$

$$A_{+}=\lim_{x\to+\infty}\frac{f(x)}{x}$$

$$B_{+}=\lim_{x\to+\infty}\left(f(x)-A_{+}x\right)$$

jeśli obie pow. granice istnieją i są skończone, to

$$y=A_{+}x+B_{+}$$

jest asymptotą ukośną w $+\infty$

(poziomą, jeśli $A_{+}=0$)

podobnie w $-\infty$:

$+ \to -$

$y=x+1$ jest asymptotą ukośną w $+\infty$ i też w $-\infty$

c) $f(x)=\dfrac{x-3}{\sqrt{x^2-9}}$ — ciągła

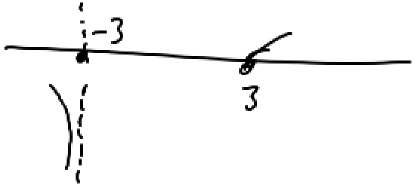
D: $x^2-9>0$

$x^2>9$

$x>3 \lor x<-3$

$\lim\limits_{x\to -3^-}\left(\dfrac{x-3}{\sqrt{x^2-9}}\right)\overset{\left[\frac{-6}{0^+}\right]}{=}-\infty \longrightarrow x=-3$ jest asymptotą pionową lewostronną

$\lim\limits_{x\to 3^+}\dfrac{x-3}{\sqrt{x^2-9}}\overset{\left[\frac{0}{0}\right]}{=}\lim\limits_{x\to 3^+}\left(\dfrac{\sqrt{x-3}\cdot\sqrt{x-3}}{\sqrt{(x-3)(x+3)}}\right)=\lim\limits_{x\to 3^+}\dfrac{\sqrt{x-3}}{\sqrt{x+3}}=0$ nie ma as. pionowej $x=3$



$$f(x) = \frac{x-3}{\sqrt{x^2-9}}$$

$$\sqrt{x^2} = |x|$$

$$A = \lim_{x\to-\infty} \frac{f(x)}{x}$$

$$B = \lim_{x\to-\infty} \left(f(x) - Ax\right)$$

$$\Downarrow$$

$$\lim_{x\to-\infty} \left(f(x) - (Ax+B)\right) = 0$$

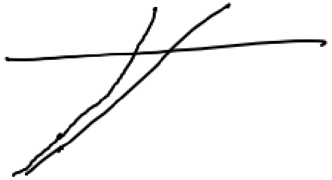
ograniczymy się do znalezienia asymptoty ukośnej w $-\infty$

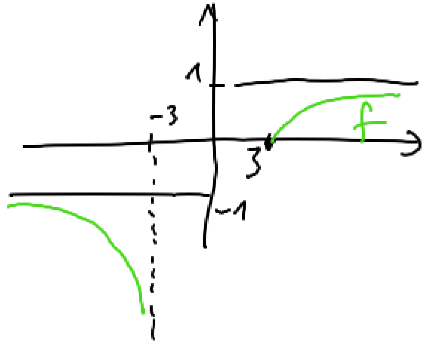
$$A = \lim_{x\to-\infty} \frac{x-3}{\sqrt{x^2-9}\cdot x} = \lim_{x\to-\infty} \frac{1-\overbrace{\frac{3}{x}}^{\to 0}}{\underbrace{\sqrt{x^2-9}}_{\to\infty}} = \frac{1}{\infty} = 0$$

$$B = \lim_{x\to-\infty} \left(\frac{x-3}{\sqrt{x^2-9}} - 0\cdot x\right) = \lim_{x\to-\infty} \frac{x\left(1-\overbrace{\frac{3}{x}}^{\to 0}\right)}{|x|\sqrt{1-\underbrace{\frac{9}{x^2}}_{\to 0}}} = -1$$

$\frac{x}{|x|} = -1$ (dla $x<0$)

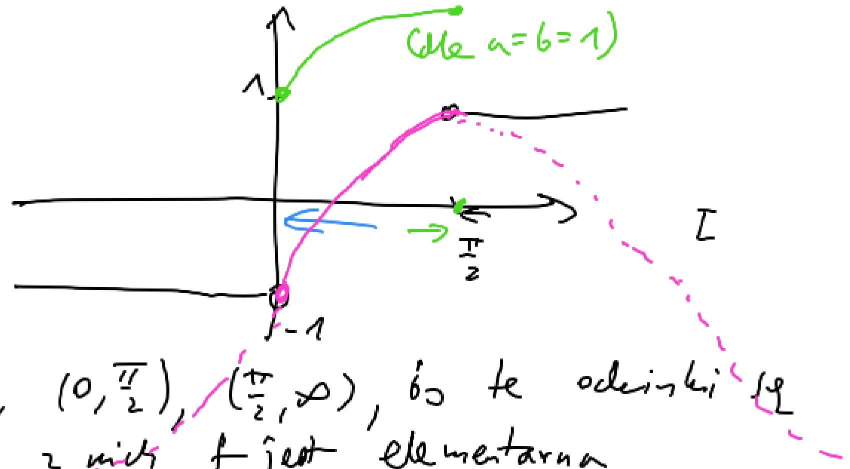
$y = 0x - 1 = -1$ jest asymptotą poziomą w $-\infty$





37 a Dobrać a, b tak, by f – ciągła:

$$f(x) = \begin{cases} -1 & \text{dla } x < 0 \\ a + b\sin x & \text{dla } 0 \le x \le \frac{\pi}{2} \\ 1 & \text{dla } x > \frac{\pi}{2} \end{cases}$$



$$\begin{cases} f(-3) = -1 \\ f(1) = a + b\sin 1 \end{cases}$$

f jest ciągła na $(-\infty, 0)$, $(0, \frac{\pi}{2})$, $(\frac{\pi}{2}, \infty)$, bo te odcinki są otwarte i na każdym z nich f jest elementarna

żeby f była ciągła w 0, musi zachodzić

$$\underbrace{\lim_{x \to 0^-} \underbrace{f(x)}_{=-1}}_{= -1} = \underbrace{f(0)}_{= a} = \underbrace{\lim_{x \to 0^+} \underbrace{f(x)}_{a + b\sin x \text{ (dla } x \to 0^+)}}_{= a}$$

$$\boxed{a = -1}$$

żeby f była ciągła w $\frac{\pi}{2}$, musi zachodzić

$$\underbrace{\lim_{x \to \frac{\pi}{2}^-} f(x)}_{a+b} = \underbrace{f\left(\frac{\pi}{2}\right)}_{a+b} = \underbrace{\lim_{x \to \frac{\pi}{2}^+} \underbrace{f(x)}_{=1}}_{=1}$$

$a + b = 1$

$-1 + b = 1$

$\underline{b = 2}$

Odp. $a = -1$, $b = 2$.

$\boxed{\text{ZADANIE 32 – KARTKÓWKA 3 XII}}$