43 a r-nie stycznej do wykr. $f(x)=x^4-2x+5$, która jest $\parallel$: $y=2x+3$

$$f'(x_0)=2$$

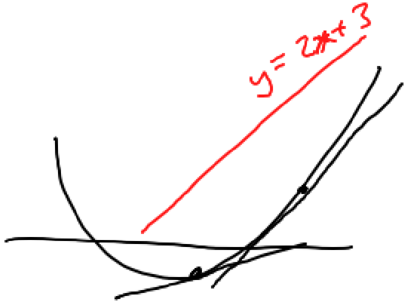
$$f'(x_0)=4x_0^3-2$$

$$4x_0^3-2=2$$

$$4x_0^3=4$$

$$x_0=1$$

$$y=2(x-1)+f(1)=2x-2+1-2+5=2x+2$$



r-nie stycznej do wykr. f w $(x_0,f(x_0))$

$$y-f(x_0)=f'(x_0)(x-x_0)$$

c) $f(x) = x \ln x$ styczna jest $\perp$ do : $2x + 6y - 1 = 0$

$6y = -2x + 1$

$y = -\frac{1}{3}x + \frac{1}{6}$

$a \cdot \left(-\frac{1}{3}\right) = -1$

$a = 3$

$f'(x_0) = 1 \cdot \ln x_0 + x_0 \frac{1}{x_0} = \ln x_0 + 1$

$\ln x_0 + 1 = 3$

$\ln x_0 = 2 \quad x_0 > 0$

$x_0 = e^2$

$y = 3(x - e^2) + e^2 \cdot \ln e^2 = 3x - 3e^2 + 2e^2 = 3x - e^2$

$$\left\{ \begin{array}{l} l_1 : y = a_1 x + b_1 \\ l_2 : y = a_2 x + b_2 \\ \text{Fakt.} \\ l_1 \perp l_2 \Leftrightarrow a_1 a_2 = -1 \end{array} \right.$$

43e $f(x) = \sin 2x - \cos 3x$ w punkcie przecięcia wykresu z osią Oy

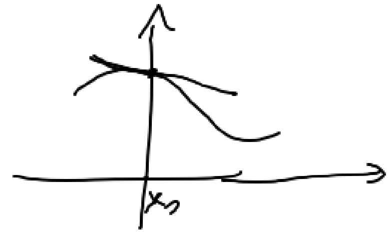
$$f'(x) = \cos(2x)\,2 + \sin(3x)\cdot 3$$

$$(0, f(0))$$

$$f(0) = 0 - 1 = -1$$

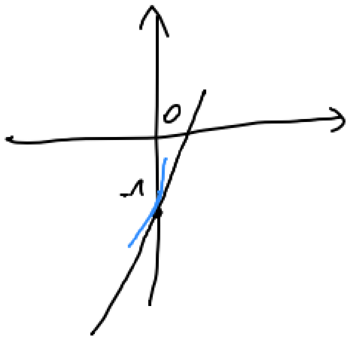
$$f'(0) = 2\cdot 1 + 3\cdot 0 = 2$$

$$y = 2(x-0) + (-1) = 2x - 1$$



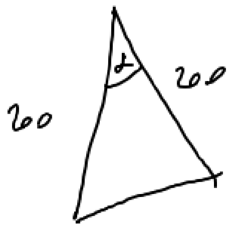
$$y - f(x_0) = f'(x_0)(x - x_0)$$

równanie stycznej do wykresu f w p. $(x_0, f(x_0))$



$y = 2x - 1$

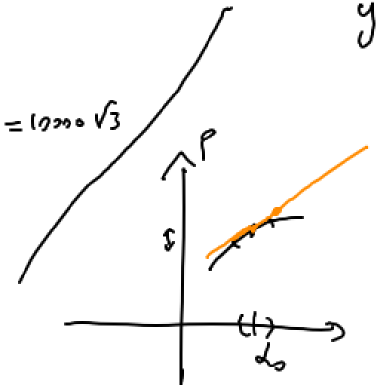
44a



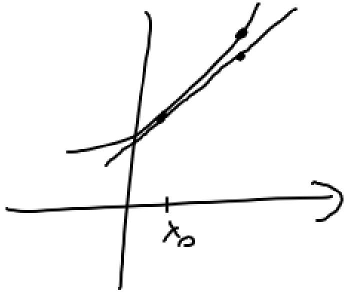
$\alpha = \overbrace{\frac{\pi}{3}}^{\alpha_0} \pm 0.01$

$P_\Delta(\alpha) = \frac{1}{2} \cdot 200 \cdot 200 \cdot \sin\alpha = 20\,000 \cdot \frac{\sqrt{3}}{2} = 10000\sqrt{3}$

$|\Delta P_\Delta| \approx |P'_\Delta(\alpha_0)| \cdot |\Delta\alpha|$



$y - f(x_0) = f'(x_0)(x - x_0)$



$P'_\Delta(\alpha) = \frac{1}{2} \cdot 200 \cdot 200 \cdot \cos\alpha$

$|\Delta P_\Delta| \approx \left|\frac{1}{2} \cdot 200 \cdot 200 \cdot \cos\frac{\pi}{3}\right| \cdot |0.01| = 10000 \cdot \frac{1}{100} = 100$

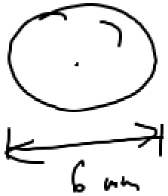
$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x - x_0)}$

wartość f. liniowej, której wykres jest prostą styczną do wykresu f

dla $x \approx x_0$

44. b) $d = \underbrace{6\text{ mm}}_{d_0} \pm \underbrace{0.01\text{ mm}}_{\Delta d}$

kulka



$\left|\frac{\Delta d}{d_0}\right| = \frac{0.01}{6} = \frac{1}{600}$

$\Delta d = d - d_0$

$d = d_0 \pm \Delta d$

d_0 – zmierzona wartość

Δd – błąd bezwzgl. pomiaru

$$V(d) = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3$$

$$V'(d) = \frac{4}{3}\pi \cdot \left(\left(\frac{1}{2}d\right)^3\right)' =$$

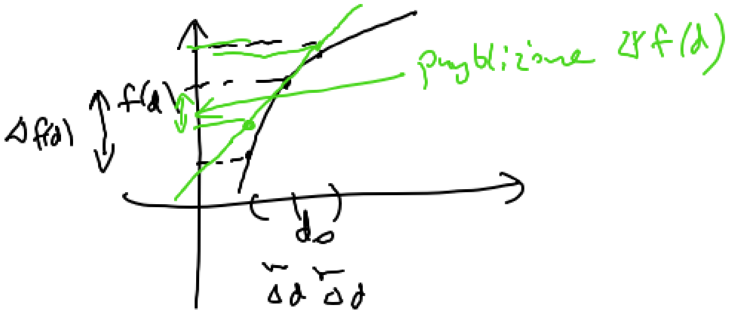
$$= \frac{\cancel{4}}{\cancel{3}}\pi \frac{1}{\cancel{8}}\cancel{3}d^2 = \frac{1}{2}\pi d^2$$

$$f(d) - f(d_0) = \Delta f(d) \approx f'(d_0)\cdot\Delta d$$

$\underbrace{\Delta f(d)}$ – błąd bezwzgl. przy obl. $f(d)$

$$|\Delta f(d)| \approx |f'(d_0)|\cdot|\Delta d|$$

$$\Delta V(d) = \frac{1}{\cancel{2}}\pi\,\cancel{36}^{\cancel{18}\,9}\cdot\frac{1}{\cancel{100}_{50}} = \frac{9}{50}\pi \quad [\text{mm}^3]$$



$$\left|\frac{\Delta V(d)}{V(d_0)}\right| \approx \frac{\frac{9}{50}\pi}{\frac{4}{3}\pi\frac{1}{8}\cdot 6^3} = \frac{\cancel{9}^{3}}{\cancel{50}_{25}}\cdot\frac{3}{\cancel{4}}\cdot\frac{\cancel{8}}{\cancel{6\cdot6\cdot6}_{8}} = \frac{1}{200}$$

15

reguła de l'Hôspitala:

Mamy $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$ typu $\frac{0}{0}$ lub $\frac{\pm\infty}{\pm\infty}, \frac{\mp\infty}{\pm\infty}$

Załóżmy, że istnieje $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$. Wtedy $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$ też istnieje

i $= \lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$.

Zapis:

$$\lim_{x\to x_0} \frac{f(\cancel{x_0})}{g(x)} \overset{H}{\underset{0/0}{=}} \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = \dots$$

jeśli istnieje

„$\lim\limits_{x\to x_0}$" może wystąpić przez $\lim\limits_{x\to x_0^-}$, $\lim\limits_{x\to x_0^+}$, $\lim\limits_{x\to\infty}$, $\lim\limits_{x\to-\infty}$.

15

a) $\lim_{x\to\infty} \frac{\ln(2^x+1)}{x} \overset{H}{\underset{\left(\frac{\infty}{\infty}\right)}{=}} \lim_{x\to\infty} \frac{\frac{1}{2^x+1} \cdot 2^x \ln(2)}{1} = \lim_{x\to\infty} \frac{2^x \ln(2)}{2^x+1} \overset{(*)}{=}$

$= \lim_{x\to\infty} \frac{\cancel{2^x} \ln(2)}{\cancel{2^x}\left(1+\frac{1}{2^x}\right)} = \frac{\ln(2)}{1} = \ln(2)$

Drugi sposób: $\lim_{x\to\infty} \frac{2^x \ln 2}{2^x+1} \overset{H}{\underset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{\cancel{2^x}(\ln 2)^2}{\cancel{2^x}\ln 2} = \ln 2$

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

b)

$$\lim_{x\to 1} \frac{\ln\left(\sin\left(\frac{\pi}{2}x\right)\right)}{\ln x} \overset{H}{\underset{\left(\frac{0}{0}\right)}{=}} \lim_{x\to 1} \frac{\frac{1}{\sin\left(\frac{\pi}{2}x\right)} \cos\left(\frac{\pi}{2}x\right) \frac{\pi}{2}}{\frac{1}{x}} =$$

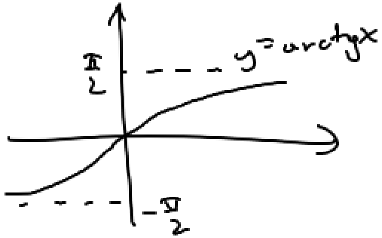
$$= \lim_{x\to 1} \frac{\frac{\pi}{2} \operatorname{ctg}\left(\frac{\pi}{2}x\right)}{\frac{1}{x}} = \frac{\pi}{2}\, 0 = 0$$

$$\left\{ \begin{array}{l} (\ln x)' = \frac{1}{x} \\ (f(g(x)))' = f'(g(x))g'(x) \end{array} \right.$$

c) $\lim\limits_{x\to 0} \dfrac{x-\operatorname{arctg}x}{x^2} \overset{H}{\underset{\left(\frac{0}{0}\right)}{=}} \lim\limits_{x\to 0}\dfrac{(x-\operatorname{arctg}x)'}{(x^2)'} =$

$$= \lim_{x\to 0}\frac{1-\frac{1}{1+x^2}}{2x} = \lim_{x\to 0}\frac{\frac{1+x^2}{1+x^2}-\frac{1}{1+x^2}}{2x} = \lim\frac{\frac{1+x^2-1}{1+x^2}}{2x} =$$

$$= \lim\frac{\frac{x^{\cancel{2}}}{1+x^2}}{2\cancel{x}} = \lim\frac{\overset{\to 0}{x}}{\underset{\to 0}{(1+x^2)2}} = 0$$



$$(\operatorname{arctg}x)' = \frac{1}{1+x^2}$$

$$(x^n)' = nx^{n-1}$$

d) $\lim\limits_{x \to 1} \frac{x^{10} - 10x + 9}{x^5 - 5x + 4} \overset{H}{\underset{\frac{0}{0}}{=}} \lim\limits_{x \to 1} \frac{(x^{10} - 10x + 9)'}{(x^5 - 5x + 4)'} = \lim\limits_{x \to 1} \frac{10x^9 - 10}{5x^4 - 5} \overset{H}{\underset{\frac{0}{0}}{=}} \lim\limits_{x \to 1} \frac{90x^8}{20x^3} = \frac{90}{20} =$

$= 4{,}5$

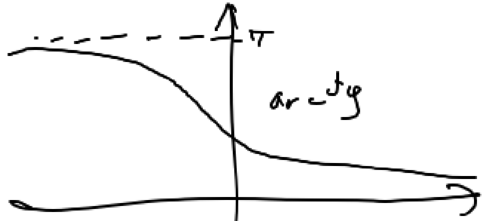
e) $\lim\limits_{x \to 0} \frac{\ln \cos x}{\ln \cos 3x} \overset{H}{\underset{(\frac{0}{0})}{=}} \lim\limits_{x \to 0} \frac{(\ln \cos x)'}{(\ln \cos 3x)'} = \lim\limits_{x \to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{\frac{1}{\cos 3x} \cdot (-\sin 3x) \cdot 3}$

aufgrun. gesetze

$= \lim\limits_{x \to 0} \frac{\frac{1}{\cos x}}{\frac{1}{\cos 3x}} \cdot \lim\limits_{x \to 0} \frac{+\sin x}{(+\sin 3x) \cdot 3} \overset{H}{\underset{[\frac{0}{0}]}{=}}$

$= \lim\limits_{x \to 0} \frac{\cos x}{3 \cos 3x \cdot 3} = \lim\limits_{x \to 0} \frac{\cos x}{9 \cdot \cos 3x} = \frac{1}{9}$

f) $\lim_{x\to\infty} (\sqrt{x}\ \operatorname{arcctg} x) = \lim_{x\to\infty} \frac{\operatorname{arcctg} x}{\frac{1}{\sqrt{x}}} =$



$$= \lim_{x\to\infty} \frac{\operatorname{arcctg} x}{x^{-\frac{1}{2}}} \overset{H}{\underset{\frac{0}{0}}{=}} \lim_{x\to\infty} \frac{\frac{-1}{1+x^2}}{-\frac{1}{2} x^{-\frac{3}{2}}} = \lim_{x\to\infty} \frac{-1}{-\frac{1}{2} x^{-\frac{3}{2}} (1+x^2)} =$$

$$a \cdot b = \frac{a}{\frac{1}{b}} = \frac{b}{\frac{1}{a}}$$

$$\underset{\downarrow \infty}{a}^{\to\infty} \quad \underset{\downarrow 0+}{b}^{\to 0} \quad \underset{\downarrow \infty}{\frac{1}{b}} \quad \underset{\downarrow 0}{\frac{1}{a}}$$

$$= \lim_{x\to\infty} \frac{1}{\underbrace{\frac{1}{2} x^{-\frac{3}{2}}}_{\to 0} + \underbrace{\frac{1}{2} x^{\frac{1}{2}}}_{\to \infty}} = \cancel{\lim_{x\to\infty} \frac{1}{\frac{1}{2}}} \quad 0$$

$$(\operatorname{arcctg} x)' = \frac{-1}{1+x^2}$$

g) $\lim_{x\to 0^+} (x \ln x) = \cancel{\lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}}}$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \left[\frac{-\infty}{\infty}\right] \overset{H}{=} \lim \frac{\frac{1}{x}}{-1x^{-2}} = \lim \frac{x^{-1}}{-1x^{-2}} = \lim_{x\to 0^+} x^{-1} \cdot \frac{1}{-x^{-2}} =$$

$$= \lim_{x\to 0^+} x^{-1} \cdot (-x^2) = \lim_{x\to 0^+} -x = 0$$