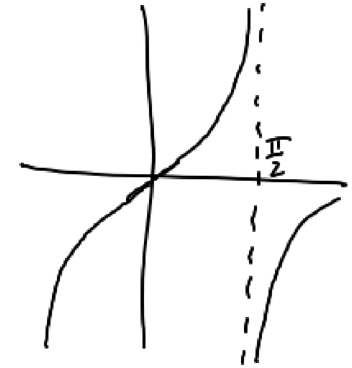
45 h)

$$\lim_{x\to\pi^-} \underbrace{(x-\pi)}_{0}\,\underbrace{\operatorname{tg}\frac{x}{2}}_{\infty} = \lim_{x\to\pi^-}\frac{(x-\pi)\,\overbrace{\sin\frac{x}{2}}^{\to 1}}{\cos\frac{x}{2}} =$$



$$= 1\cdot \lim_{x\to\pi^-}\frac{x-\pi}{\cos\frac{x}{2}} \overset{H}{\underset{\left(\frac{0}{0}\right)}{=}} \lim_{x\to\pi^-}\frac{(x-\pi)'}{\left(\cos\frac{x}{2}\right)'} =$$

$$= \lim_{x\to\pi^-}\frac{1}{\underbrace{-\sin\underbrace{\frac{x}{2}}_{\frac{\pi}{2}}}_{1}\cdot\frac{1}{2}} = \frac{1}{-1\cdot\frac{1}{2}} = -2$$

i) $$\lim_{x\to 0^+}\left(\underbrace{\frac{1}{1-\cos x}}_{0^+}-\underbrace{\frac{1}{x^2}}_{0^+}\right)=\lim_{x\to 0^+}\left(\frac{x^2-1+\cos x}{x^2(1-\cos x)}\right)\overset{H}{\underset{\frac{0}{0}}{=}}\lim_{x\to 0^+}\frac{2x-\sin x}{2x(1-\cos x)+x^2(+\sin x)}$$

$$=\lim_{x\to 0^+}\frac{2x-\sin x}{2x-2x\cos x+x^2\sin x}\overset{H}{=}\lim_{x\to 0^+}\frac{2-\cos x}{\underbrace{2-2\cos x}_{>0}+\underbrace{2x\sin x}_{>0}+\underbrace{2x\sin x}_{>0}+\underbrace{x^2\cdot\cos x}_{>0}}=\frac{2-1}{(2-2)^+}=+\infty$$

k)

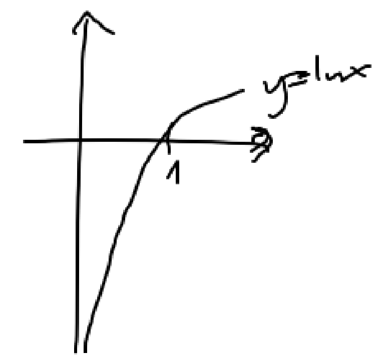
$$\lim_{x\to 1}\left(\frac{1}{\ln x}+\frac{1}{1-x}\right)=\lim_{x\to 1}\frac{1-x+\ln x}{(1-x)\cdot\ln x}\overset{H}{\underset{\left(\frac{0}{0}\right)}{=}}\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\ln x+(1-x)\cdot\frac{1}{x}}=$$

$$=\lim_{x\to 1}\frac{-1+\frac{1}{x}}{-\ln x+\frac{1}{x}-1}\overset{H}{\underset{\left(\frac{0}{0}\right)}{=}}\lim_{x\to 1}\frac{-\frac{1}{x^2}}{-\frac{1}{x}-\frac{1}{x^2}}=\frac{-1}{-2}=\frac{1}{2}$$

e) $\lim\limits_{x\to 0^+}(-\ln x)^x = \lim\limits_{x\to 0^+} e^{\ln(-\ln x)\cdot x} = (*)$

$a^b = (e^{\ln a})^b = e^{b\cdot \ln a}$

$a\cdot b = \dfrac{a}{\frac{1}{b}}$



Policzmy

$$\lim_{x\to 0^+} \underbrace{\ln(\underbrace{-\ln x}_{\infty})}_{\infty}\cdot \underbrace{x}_{0} = \lim_{x\to 0^+} \frac{\ln(-\ln x)}{\frac{1}{x}} \overset{H}{\underset{\left(\frac{\infty}{\infty}\right)}{=}}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{-\ln x}\cdot(-\ln x)'}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\frac{1}{-\ln x}\cdot\left(\cancel{-}\cancel{\frac{1}{x}}\right)}{\cancel{-}\frac{1}{x^{\cancel{2}}}} = \frac{\frac{1}{\infty}}{\infty} = \frac{0}{\infty} = 0$$

Wobec tego

$(*) = e^0 = 1$

(m) $\lim\limits_{x\to\infty}\left(\frac{2}{\pi}\operatorname{arctg}x\right)^x = \lim\limits_{x\to\infty} e^{x\cdot\ln\left(\frac{2}{\pi}\operatorname{arctg}x\right)} = e^{-\frac{2}{\pi}}$

$a^b = e^{b\ln a}$

$(\operatorname{arctg}x)' = \frac{1}{1+x^2}$

$$\lim_{x\to\infty}\left(x\ln\left(\frac{2}{\pi}\operatorname{arctg}x\right)\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{2}{\pi}\operatorname{arctg}x\right)}{\frac{1}{x}} \overset{H}{\underset{\left(\frac{0}{0}\right)}{=}}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{\cancel{\frac{2}{\pi}}\operatorname{arctg}x}\cdot\cancel{\frac{2}{\pi}}\cdot\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty}\frac{-1}{\operatorname{arctg}x}\,\frac{x^2}{1+x^2} =$$

$$= \lim_{x\to\infty}\frac{-1}{\underbrace{\operatorname{arctg}x}_{\to\frac{\pi}{2}}}\,\frac{1}{1+\underbrace{\frac{1}{x^2}}_{\to 0}} = -\frac{2}{\pi}$$

46 Znaleźć przedziały monotoniczności funkcji:

$-\infty \le a < b \le \infty$

Fakt. Jeśli f jest różniczkowalna na (a,b) ~~oraz~~ :

1) oraz $f' > 0$ na (a,b), to f jest rosnąca na (a,b)

2) oraz $f' < 0$ na (a,b), to — " — malejąca na (a,b)

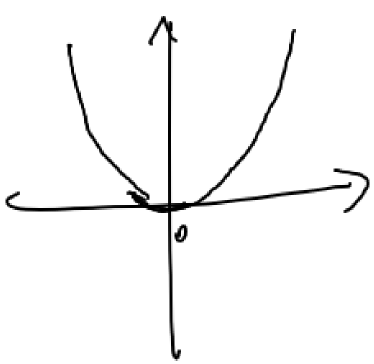
· Jeśli f jest różniczkowalna na (a,b) i ciągła na $[a,b]$,

1) .. oraz $f' > 0$ na (a,b), to f jest rosnąca na $[a,b]$

2) .. oraz $f' < 0$ na (a,b), to f jest malejąca na $[a,b]$

można zamienić $[a,b]$ na $(a,b]$ lub $[a,b)$

Np. $f(x) = x^2$



$f'(x) = 2x$

$f' > 0$ na $(0,\infty) \Rightarrow f$ jest rosnąca na $[0,\infty)$

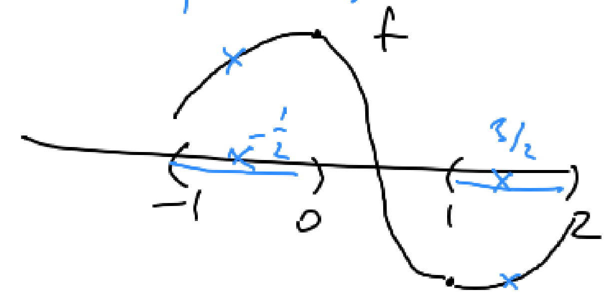
(f jest ciągła w 0)

$f' < 0$ na $(-\infty,0) \Rightarrow f$ jest malejąca na $(-\infty,0]$

Uwaga: Jeśli np. $f' > 0$ na $(-1,0) \cup (1,2)$, to f jest rosnąca na $(-1,0)$

i jest rosnąca na $(1,2)$

Ale nie jest rosnąca na $(-1,0)\cup(1,2)$



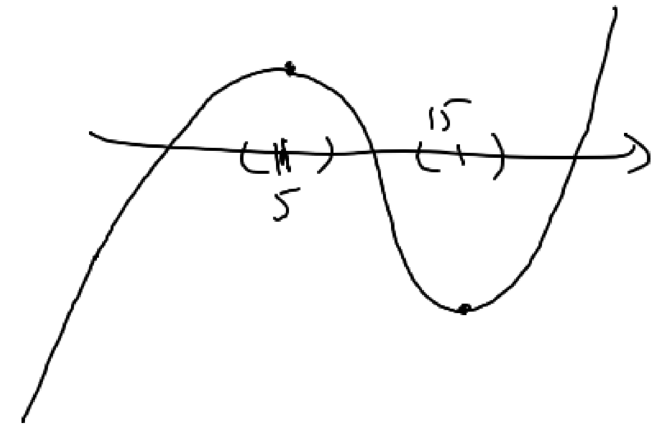
$f(-\frac{1}{2}) > f(3/2)$

$-\frac{1}{2} < 3/2$

46 a)

$$f(x) = x^3 - 30x^2 + 225x$$

$$f'(x) = 3x^2 - 60x + 225$$



$$3x^2 - 60x + 225 = 0$$

$$x^2 - 20x + 75 = 0$$

$$\Delta = 400 - 300 = 10^2$$

$$x_1 = \frac{10}{2} = 5$$

$$x_2 = \frac{30}{2} = 15$$

$f'(x) > 0$ dla $x \in (-\infty, 5) \cup (15, +\infty)$

$f'(x) < 0$ dla $x \in (5, 15)$

$f \nearrow$ na $(-\infty, 5]$

$f \nearrow$ na $[15, \infty)$

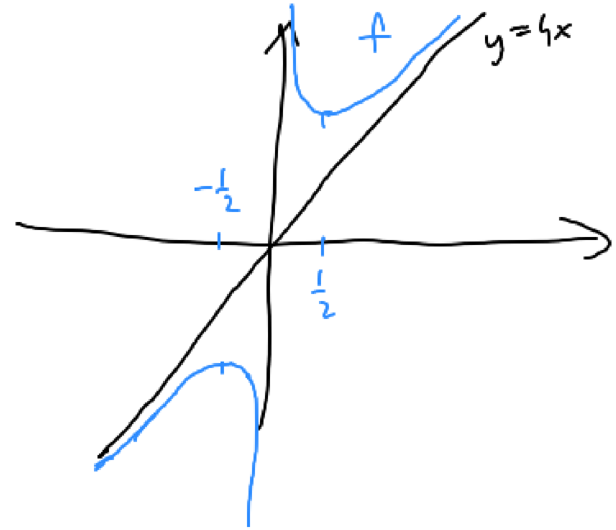
$f \searrow$

	$(-\infty, 5]$	$[5, 15]$	$[15, \infty)$
f'	+	−	+
f	$\nearrow$	$\searrow$	$\nearrow$

c) $f(x) = 4x + \frac{1}{x}$

$D_f = \mathbb{R} \setminus \{0\}$

$f'(x) = 4 - \frac{1}{x^2}$ $\qquad D_{f'} = D_f$



$f'(x) > 0 \Leftrightarrow 4 - \frac{1}{x^2} > 0$

$4 > \frac{1}{x^2} \quad \big| \cdot x^2 > 0$

$4x^2 > 1$

$x^2 > \frac{1}{4}$

$x \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$

$f'(x) < 0 \Leftrightarrow 4 - \frac{1}{x^2} < 0$

$\vdots$

$x^2 < \frac{1}{4}$

$x \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$

	$(-\infty, -\frac{1}{2})$	$(-\frac{1}{2}, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, \infty)$
f'	+	−	−	+
f	$\nearrow$	$\searrow$	$\searrow$	$\nearrow$

f) $f(x) = xe^{-3x}$ $\quad D_f = \mathbb{R}$ $\quad$ ciągła $\qquad (gh)' = g'h + gh'$

$$f'(x) = 1\cdot e^{-3x} + x\cdot e^{-3x}\cdot(-3x)' = e^{-3x}(1 + x\cdot(-3)) = \underbrace{e^{-3x}}_{>0}\underbrace{(1-3x)}_{\cancel{\geq 0}}$$

$f' > 0$ na $(-\infty, \frac{1}{3}) \Rightarrow f$ jest $\nearrow$ na $(-\infty, \frac{1}{3}]$

$f' < 0$ na $(\frac{1}{3}, \infty) \Rightarrow f$ jest $\searrow$ na $[\frac{1}{3}, \infty)$