50. całki nieoznaczone

$\int f(x)\,dx = F(x) + c$, jeśli $F'(x) = f(x)$ na jakimś przedziale

$\left(= \{F(x)+c : c \in \mathbb{R}\} \right)$

F jest funkcją pierwotną f

$$F+c \xrightarrow{\ '\ } f, \qquad f \xrightarrow{\ \int\ } F+c$$

a) $\int (x^3 + \frac{4}{x} - 3\sqrt{x})\,dx = \int x^3\,dx + 4\int \frac{1}{x}\,dx - 3\int \underbrace{\sqrt{x}}_{x^{1/2}}\,dx =$

$= \frac{1}{4}x^4 + 4\ln x - 3 \cdot \frac{2}{3}\sqrt{x^3} + C$

ogólnie $\ln|x|$, ale tutaj $x > 0$, bo występuje $\sqrt{x}$ i $\frac{1}{x}$

b) $\int \frac{1-x}{1+\sqrt{x}}\,dx = \int \frac{(1-\sqrt{x})\cancel{(1+\sqrt{x})}}{\cancel{1+\sqrt{x}}} = \int (1-\sqrt{x})\,dx =$

$= \int (1 - x^{\frac{1}{2}})\,dx = x - \frac{2x\sqrt{x}}{3} + C$

$(x^n)' = nx^{n-1}$

$(x^{n+1})' = (n+1)x^n$

$\left(\frac{x^{n+1}}{n+1}\right)' = x^n$

$\Downarrow$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1}$$

$\int (f(x) + g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$

$\int cf(x)\,dx = c\int f(x)\,dx$

$\int \frac{1}{x}\,dx = \ln|x| + c$

c) $\int \frac{x^4}{x^2+1}\,dx = \int x^2 - 1 + \frac{1}{x^2+1}\,dx = \int x^2\,dx - \int 1\,dx + \int \frac{1}{x^2+1}\,dx =$

$= \frac{x^3}{3} - x + \operatorname{arctg}(x) + C$

$\int \frac{1}{x^2+1}\,dx = \operatorname{arctg} x + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

d) $\int \frac{\cos 2x}{\cos x - \sin x}\,dx = \int \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x} = \int \frac{\cancel{(\cos x - \sin x)}(\cos x + \sin x)}{\cancel{\cos x - \sin x}} =$

$= \int \cos x + \int \sin x = \sin x - \cos x + C$

e) $\int \frac{x^3 + \sqrt[3]{x^2} - 1}{\sqrt{x}} dx =$

$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + c, & n \neq -1 \\ \ln|x| + c, & n = -1 \end{cases}$

$= \int \frac{x^3 + x^{2/3} - 1}{x^{1/2}} dx = \int \left(x^{\frac{5}{2}} + x^{\frac{1}{6}} - x^{-\frac{1}{2}} \right) dx =$

$= \frac{x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{x^{\frac{1}{6}+1}}{\frac{1}{6}+1} - \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = \underline{\frac{2}{7} x^{7/2} + \frac{6}{7} x^{7/6} - 2x^{\frac{1}{2}} + C}$

f) $\int e^{-x} 3^{2x} dx = \int e^{-x} \cdot 3^{2x} dx = \int \frac{1}{e^x} \cdot 3^{2x} = \int \frac{3^{2x}}{e^x} =$

$\int a^x dx = \frac{a^x}{\ln a} + c$

$(a > 0, a \neq 1)$

$= \int \left(\frac{9}{e} \right)^x = \frac{\left(\frac{9}{e} \right)^x}{\ln \left(\frac{9}{e} \right)} + C$

Z2

$$\int f'(x)\cdot g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$$

a) $$\int x e^{-3x}\,dx = \int \underbrace{\left(\frac{e^{-3x}}{-3}\right)'}_{f(x)} \cdot \underbrace{x}_{g(x)}\,dx = \frac{e^{-3x}}{-3}x - \int \frac{e^{-3x}}{-3}\cdot(x)'\,dx =$$

$$= -\frac{1}{3}e^{-3x}x + \frac{1}{3}\int e^{-3x}\,dx = -\frac{1}{3}e^{-3x}x + \frac{1}{3}\left(\frac{e^{-3x}}{-3}\right) + C$$

$$\int\left(\frac{x^2}{2}\right)' e^{-3x}\,dx = \frac{x^2}{2}e^{-3x} - \int \frac{x^2}{2}\left(e^{-3x}\right)'dx =$$

$$= \frac{x^2}{2}e^{-3x} - \underbrace{\int \frac{x^2}{2}e^{-3x}\cdot(-3)\,dx}$$ ⟵ nic nie daje

b) $$\int (x+1)^2 e^x\,dx = \int (e^x)'(x+1)^2\,dx = e^x\cdot(x+1)^2 - \int e^x\cdot 2(x+1)\,dx =$$

$$= e^x\cdot(x+1)^2 - 2\cdot\left[\int (e^x)'\cdot(x+1)\,dx\right] = e^x\cdot(x+1)^2 - 2\left[e^x\cdot(x+1) - \int e^x\,dx\right] =$$

$$= e^x(x+1)^2 - 2e^x\cdot(x+1) + 2e^x + C$$

c) $\int \sqrt{x} \cdot \operatorname{arctg}\sqrt{x}\, dx = \int \left(\frac{2\sqrt{x^3}}{3}\right)' \cdot \operatorname{arctg}\sqrt{x}\, dx =$

$\int f'g = fg - \int fg'$

$= \frac{2x\sqrt{x}}{3} \cdot \operatorname{arctg}\sqrt{x} - \int \frac{\cancel{2}}{3} \cdot x\cancel{\sqrt{x}} \cdot \left(\frac{1}{x+1}\right) \cdot \frac{1}{\cancel{2}\cancel{\sqrt{x}}}\, dx =$

$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + c, & n \neq -1 \\ \ln|x| + c, & n = -1 \end{cases}$

$= \frac{2}{3} x\sqrt{x}\, \operatorname{arctg}\sqrt{x} - \frac{1}{3} \int \frac{x}{x+1}\, dx =$

$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

$\left\{ \int \frac{x}{x+1}\, dx = \int \frac{x+1-1}{x+1}\, dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \int \frac{1}{x+1}\, dx = x - \ln|x+1| + C \right.$

$= \frac{2}{3} x\sqrt{x}\, \operatorname{arctg}\sqrt{x} - \frac{1}{3}\left(x - \ln|x+1|\right) + \tilde{C}$

$$\int f'g = fg - \int fg'$$

g)

$$\int \ln(x+1)\,dx = \int 1\cdot \ln(x+1)\,dx = \int (x)' \ln(x+1)\,dx =$$

$$= x\ln(x+1) - \int x\,\big(\ln(x+1)\big)'\,dx = x\ln(x+1) - \underbrace{\int x\cdot\frac{1}{x+1}\,dx}_{\text{liczyliśmy przed chwilą}}$$

h)

$$\int \arccos x\,dx = \int (x)' \arccos x\,dx = x\arccos x - \int x\,\frac{-1}{\sqrt{1-x^2}}\,dx =$$

$$= x\arccos x + \int \frac{x}{\sqrt{1-x^2}}\,dx =$$

$$= x\arccos x - \sqrt{1-x^2} + C$$

$$\left(\arcsin\left(\frac{x^2}{2}\right)\right)' = \frac{1}{\sqrt{1-\left(\frac{x^2}{2}\right)^2}}\cdot \cancel{x}$$

$$\left(\sqrt{1-x^2}\right)' = \frac{1}{2\sqrt{1-x^2}}\cdot(-2x)$$

$$\boxed{\int f'g = fg - \int fg'}$$

e)

$$\int x^2 \sin x \, dx = \int (-\cos x)' x^2 \, dx = -\cos x \cdot x^2 - \left(\int -\cos x \cdot 2x \, dx\right) =$$

$$= -x^2 \cos x - \left(\int (-\sin x)' \cdot 2x \, dx\right) =$$

$$= -x^2 \cos x - \left(-2x \sin x - \int -\sin x \cdot 2\right) =$$

$$= -x^2 \cos x - \left(-2x \sin x - 2 \cdot \cos x\right) + C$$

$\boxed{\int f'g = fg - \int fg'}$

j)

$$\underbrace{\int \sin x \cdot \sin 3x \, dx} = \int (-\cos x)' \sin 3x \, dx = -\cos x \sin 3x - \int (-\cos x) \cdot 3 \cos 3x \, dx =$$

$$= -\cos x \sin 3x + 3 \int \cos x \cos 3x \, dx = -\cos x \sin 3x + 3 \int (\sin x)' \cos 3x \, dx =$$

$$= -\cos x \sin 3x + 3 \left[\sin x \cos 3x - \int \sin x \cdot (-3) \sin 3x \, dx \right] =$$

$$= -\cos x \sin 3x + 3 \sin x \cos 3x + 9 \underbrace{\int \sin x \sin 3x \, dx}$$

$$-8 \int \sin x \sin 3x \, dx = -\cos x \sin 3x + 3 \sin x \cos 3x + C$$

$$\int \sin x \sin 3x \, dx = \frac{1}{8} \cos x \sin 3x - \frac{3}{8} \sin x \cos 3x \underbrace{\boxed{-\frac{1}{8} C}}_{+\tilde{C}}$$

$$\int \quad e^{ax} \cos bx \qquad e^{ax} \sin bx \qquad \underset{\sin}{\cos ax} \; \underset{\cos}{\sin bx}$$

$$\sin x \sin 3x = \frac{e^{ix}-e^{-ix}}{2i}\,\frac{e^{i3x}-e^{-3ix}}{2i} = \frac{e^{4ix}-e^{-2ix}-e^{2ix}+e^{-4ix}}{-4} =$$

50, 52

[illegible]

$$= -\frac{1}{2}\,\frac{e^{4ix}+e^{-4ix}}{2} + \frac{1}{2}\,\frac{e^{-2ix}+e^{2ix}}{2} = -\frac{1}{2}\cos 4x + \frac{1}{2}\cos 2x$$

$$\int \sin x \sin 3x \, dx = \int \left(-\frac{1}{2}\cos 4x + \frac{1}{2}\cos 2x\right) dx = -\frac{1}{2}\frac{1}{4}\sin 4x + \frac{1}{2}\frac{1}{2}\sin 2x + C$$

$$\int f'g = fg - \int fg'$$

$$\int e^{2x}\sin x\,dx = \int e^{2x}(-\cos x)'\,dx = e^{2x}(-\cos x) + \int 2e^{2x}\cos x\,dx =$$

$$= -e^{2x}\cos x + 2\int e^{2x}(\sin x)'\,dx = e^{-2x}\cos x + 2\left[e^{2x}\sin x - \int 2e^{2x}\sin x\,dx\right] =$$

$$= e^{-2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$$

$$5\int e^{2x}\sin x\,dx = e^{-2x}\cos x + 2e^{2x}\sin x + C$$

$$\Rightarrow \boxed{\int e^{2x}\sin x\,dx = \frac{1}{5}e^{-2x}\cos x + \frac{2}{5}e^{2x}\sin x + \frac{C}{5}}$$