53

a) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx = \left| \begin{array}{l} t=\sqrt{x} \\ dt = \frac{1}{2\sqrt{x}}\,dx \end{array} \right.$

$t(x)=\sqrt{x}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{x}}$

$2\,dt = \frac{dx}{\sqrt{x}}$

$\int \cos t \cdot 2\,dt = 2\sin t + C = 2\sin\sqrt{x} + C$

$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{A(t+1)+B(t-1)}{t^2-1}$

$1 = A(t+1)+B(t-1)$

$t=1:\quad 1=2A \quad A=\frac{1}{2}$

$t=-1:\quad 1=-2B \quad B=-\frac{1}{2}$

b) $\int \frac{\sqrt{1+4x}}{x}\,dx = \left| \begin{array}{l} y=1+4x \\ dy = 4\,dx \end{array} \right| = \int \frac{\sqrt{y}}{\frac{y-1}{4}}\frac{1}{4}\,dy = \int \frac{\sqrt{y}}{y-1}\,dy = \left| \begin{array}{l} t=\sqrt{y} \\ dt = \frac{1}{2\sqrt{y}}\,dy \\ dy = 2t\,dt \end{array} \right| =$

$y-1=4x$

$x=\frac{y-1}{4}$

$t^2=y$

$= \int \frac{t}{t^2-1}\,2t\,dt = 2\int \frac{t^2}{t^2-1}\,dt = 2\int \frac{t^2-1+1}{t^2-1}\,dt = 2\int \left(1+\frac{1}{t^2-1}\right)dt =$

$= 2t + \int \left(\frac{\frac{1}{2}}{t-1} - \frac{\frac{1}{2}}{t+1}\right)dt = 2t + \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| + C =$

$= 2\sqrt{1+4x} + \frac{1}{2}\ln\left|\sqrt{1+4x}-1\right| - \frac{1}{2}\ln\left|\sqrt{1+4x}+1\right| + C$

c) $\int \frac{\cos x}{\sqrt{1+\sin x}}\,dx$ $\quad 1+\sin x \to t$

$$\left| \begin{array}{l} t = 1+\sin x \\ dt = \cos x\,dx \\ dt = \cos dx \end{array} \right.$$

$$\int \frac{dt}{\sqrt{t}} = 1\cdot\int t^{-\frac{1}{2}}\cdot dt = \frac{1}{-\frac{1}{2}+1}\cdot t^{\frac{1}{2}} \overset{+C}{\checkmark} = \frac{1}{\frac{1}{2}} t^{\frac{1}{2}} \overset{+C}{\checkmark} = 2\cdot t^{\frac{1}{2}} \overset{+C}{\checkmark} =$$

$$= 2(1+\sin x)^{\frac{1}{2}} + C$$

d) $\int x\sin(x^2+4)\,dx =$ $\quad x^2+4 \to t$

$$\left| \begin{array}{l} t = x^2+4 \\ dt = 2x\,dx \\ \frac{dt}{2} = x\,dx \end{array} \right| = \frac{1}{2}\int \sin t\,dt = -\frac{1}{2}\cos t + C =$$

$$= -\frac{1}{2}\cos(x^2+4) + C$$

j) $\int \frac{e^x}{e^{2x}+1}\,dx = \left| \begin{matrix} t = e^x \\ dt = e^x dx \end{matrix} \right. = \int \frac{dt}{t^2+1} = \int \frac{1}{t^2+1}\,dt =$

$= \operatorname{arctg} t + C = \operatorname{arctg}(e^x) + C$

---

e) $\operatorname{cth} x = \frac{\cosh x}{\sinh x}$ $\qquad \cosh x = \frac{e^x+e^{-x}}{2}$ $\qquad \sinh x = \frac{e^x-e^{-x}}{2}$

$(\cosh x)' = \sinh x$

$(\sinh x)' = \cosh x$

$\int \frac{\sinh x}{\cosh x}\,dx = \left| \begin{matrix} t = \cosh x \\ dt = \sinh x\,dx \end{matrix} \right| = \int \frac{1}{t}\,dt = \ln|t| + C = \ln|\cosh x| + C$

f) $5-3x = t$

i) $t = \ln x$

g) $t = 5x^3+1$

h) $t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}}\,dx$

$dx = 2t\,dt$

e) $t = x^2$

potem przez części

65

a) $\cancel{\int \frac{dx}{x^2+4x+29}}$

ut. pr. II rodzaju

$$\int \frac{Ax+B}{(x^2+px+q)^k}\,dx$$

$p^2-4q<0, \quad k\in\mathbb{N}$

Np.

$$\int \frac{3x+5}{\underbrace{x^2+4x+7}_{\Delta=16-28<0}}\,dx = \int \frac{\frac{3}{2}(2x+4)-1}{x^2+4x+7}\,dx = \frac{3}{2}\underbrace{\int \frac{2x+4}{x^2+4x+7}\,dx}_{} - \int \frac{dx}{x^2+4x+7}$$

$(x^2+4x+7)' = 2x+4$

$t = x^2+4x+7$

$dt = (2x+4)\,dx$

$$= \int \frac{dt}{t} = \ln|x^2+4x+7| + C$$

$$\int \frac{dx}{x^2+4x+7} = \int \frac{dx}{(x+2)^2+3} = \frac{1}{3}\int \frac{dx}{\frac{(x+2)^2}{3}+1} = \frac{1}{3}\int \frac{dx}{\left(\frac{x+2}{\sqrt{3}}\right)^2+1} = \left| \begin{array}{l} \frac{x+2}{\sqrt{3}} = y \\ \frac{dx}{\sqrt{3}} = dy \end{array} \right|$$

$dx = \sqrt{3}\,dy$

$$= \frac{1}{3}\int \frac{\sqrt{3}\,dy}{y^2+1} = \underline{\frac{\sqrt{3}}{3}\,\mathrm{arctg}\left(\frac{x+2}{\sqrt{3}}\right) + C}$$

a)

$$\int \frac{dx}{x^2+4x+29} = \int \frac{dx}{(x+2)^2+25} = \frac{1}{25}\int \frac{dx}{\left(\frac{x+2}{5}\right)^2+1} = \left| \begin{aligned} t &= \frac{x+2}{5} \\ dt &= \frac{1}{5}dx \\ 5dt &= dx \end{aligned} \right|$$

$$= \frac{1}{25}\int \frac{5\,dt}{t^2+1} = \frac{1}{5}\,\mathrm{arctg}\left(\frac{x+2}{5}\right) + C$$

b)

$$\int \frac{6x+3}{x^2+x+4}\,dx = \int \frac{3(2x+1)}{x^2+x+4}\,dx = \left| \begin{aligned} t &= x^2+x+4 \\ dt &= (2x+1)dx \end{aligned} \right| = \int \frac{3\,dt}{t} =$$

$$(x^2+x+4)' = 2x+1$$

$$= 3\ln|t| + C = 3\ln\big(\underbrace{x^2+x+4}_{>0}\big) + C$$

c) $$\int \frac{(4x+2)\,dx}{x^2-10x+29} = \int \frac{2(2x-10)+22}{x^2-10x+29}\,dx = \underbrace{2\int \frac{2x-10}{x^2-10x+29}\,dx}_{I_1} + \underbrace{22\int \frac{dx}{x^2-10x+29}}_{I_2}$$

$$(x^2-10x+29)' = 2x-10$$

$$I_1 = \int \frac{2x-10}{x^2-10x+29}\,dx = \left| \begin{array}{l} t = x^2-10x+29 \\ dt = (2x-10)\,dx \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C$$
$$= \ln(x^2-10x+29) + C$$

$$I_2 = \int \frac{dx}{x^2-10x+29} = \int \frac{dx}{(x-5)^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x-5}{2}\right)^2+1} = \left| \begin{array}{l} y = \frac{x-5}{2} \\ dy = \frac{1}{2}\,dx \\ 2\,dy = dx \end{array} \right| =$$

$$= \frac{1}{4}\int \frac{2\,dy}{y^2+1} = \frac{1}{2}\,\mathrm{arctg}\left(\frac{x-5}{2}\right) + C$$

64c

$$\int \frac{5\,dx}{(2-7x)^3} = \left| \begin{array}{l} t=2-7x \\ dt=-7\,dx \\ dx=-\frac{1}{7}\,dt \end{array} \right| = \int \frac{5\cdot\left(-\frac{1}{7}\right)dt}{t^3} =$$

$$= -\frac{5}{7}\int t^{-3}\,dt = -\frac{5}{7}\,\frac{t^{-2}}{-2} + C =$$

$$= \frac{5}{14}\,\frac{1}{(2-7x)^2} + C$$

$$\frac{A}{(x-a)^k}$$

$$\int t^n\,dt = \begin{cases} \frac{t^{n+1}}{n+1} + C, & n \neq -1 \\ \ln|t| + C, & n = -1 \end{cases}$$

66a

$$\int \frac{x+2}{x(x-2)}\,dx = \int \left(\frac{-1}{x} + \frac{2}{x-2}\right)dx = -\ln|x| + 2\ln|x-2| + C$$

$$\frac{x+2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} = \frac{A(x-2)+Bx}{x(x-2)}$$

st. licznika $<$ st. mianownika

$$Ax - 2A + Bx = x+2$$

$$\begin{cases} A+B=1 \\ -2A=2 \end{cases} \quad A=-1 \quad B=2$$

b) $$\int \frac{x^2}{x+1}\,dx = \int \left(x-1+\frac{1}{x+1}\right)dx = \frac{x^2}{2} - x + \ln|x+1| + C$$

$$\begin{array}{l} \quad x-1 \quad \text{iloraz} \\ \hline x^2 : (x+1) \\ \underline{-(x^2+x)} \\ \quad -x \\ \underline{\quad -(-x-1)} \\ \qquad 1 \quad \text{reszta} \end{array}$$

$$\frac{x^2}{x+1} = \underbrace{x-1}_{\text{iloraz}} + \frac{\overset{\text{reszta}}{1}}{x+1}$$

$$\frac{11}{3} = \underbrace{3}_{\text{iloraz}} + \frac{\overset{\text{reszta}}{2}}{3}$$

h) $$\int \frac{dx}{x(x^2-4)} = \int \left(\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}\right) dx = A\ln|x| + B\ln|x-2| + C\ln|x+2| + D$$

(stała całkowania)

$$\frac{1}{x(x^2-4)} = \frac{1}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

Kartkówki:

I	53
II	65

67

a) $\int \sin^3 x \, dx =$

$\left| \begin{array}{l} t = \sin x \\ dt = \cos x \, dx \end{array} \right|$ ← to nie jest dobry sposób

$= \int \frac{\overset{t^3}{\boxed{\sin^3 x}} \, \overset{dt}{\boxed{\cos x \, dx}}}{\cos x}$

tym nie ma co zrobić :C

$\left| \begin{array}{l} t = \cos x \\ dt = -\sin x \end{array} \right|$

$= -\int \underbrace{\sin^2 x}_{= 1 - \cos^2 x = 1 - t^2} \cdot \overset{dt}{\boxed{(-\sin x) \, dx}} = -\int (1 - t^2) \, dt = -\left(t - \frac{t^3}{3}\right) + C$

$= -\cos x + \frac{\cos^3 x}{3} + C$

Podstawienia

- $t = \sin x$
- $t = \cos x$
- $t = \operatorname{tg} x$ (gdy występują: $\operatorname{tg} x$, $\operatorname{ctg} x$, $\sin^2 x$, $\cos^2 x$)
- $t = \operatorname{tg} \frac{x}{2}$

b)

$$\int \sin^4 x \underbrace{\cos^3 x\, dx}_{\cos^2 x \boxed{\cos x\, dx}} = \left| \begin{array}{l} t = \sin x \\ dt = \cos x\, dx \end{array} \right| = \int t^4 (1-t^2)\, dt =$$

$$= \int (t^4 - t^6)\, dt = \frac{t^5}{5} - \frac{t^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$$

c) $\int \cos^4 x\, dx = \frac{3}{8} x + \frac{1}{2}\frac{1}{2} \sin 2x + \cancel{\frac{1}{8}}\frac{1}{4} \sin 4x + C$

$\left.\begin{array}{l} t = \sin x \\ t = \cos x \end{array}\right\}$ nie działają

$t = \operatorname{tg} x$ — działa, ale prowadzi do niezgrabnych rachunków

$$+\left| \begin{array}{l} \cos^2 x - \sin^2 x = \cos 2x \\ \cos^2 x + \sin^2 x = 1 \end{array} \right.$$

$2\cos^2 x = 1 + \cos 2x$

$$\boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$$

$$\underline{\cos^4 x} = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\,\frac{1+\cos 4x}{2} =$$

$$= \underline{\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x}$$