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$$\int \frac{dx}{\sin x} = \int \frac{1}{\sin(x)} \cdot \frac{\sin x}{\sin x}\, dx = \int \frac{\sin x}{\sin^2 x}\, dx =$$

$t = \sin x$
$t = \cos x$
$t = \operatorname{tg} x$
$t = \operatorname{tg} \frac{x}{2}$

$$\int \frac{\sin x}{1-(\cos x)^2}\, dx = \left| \begin{array}{l} t = \cos x \\ dt = -\sin x\, dx \\ -dt = \sin x\, dx \end{array} \right. = -\int \frac{dt}{1-t^2} = -\int \frac{1}{-(t^2-1)}\, dt =$$

$$= \int \frac{1}{t^2-1}\, dt = \int \frac{dt}{t^2-(1)^2} \underset{?}{=} \frac{1}{2} \cdot \ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{2} \ln\left|\frac{\cos x - 1}{\cos x + 1}\right| + C$$

$$\frac{1}{t^2-1} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{A(t+1)+B(t-1)}{t^2-1} = \frac{(A+B)\,t + (A-B)}{t^2-1}$$

$$\begin{cases} A+B=0 \\ A-B=1 \end{cases} \qquad A = \frac{1}{2} \qquad B = -\frac{1}{2}$$

$$\int \frac{dt}{t^2-1} = \frac{1}{2}\int \frac{1}{t-1}\, dt - \frac{1}{2}\int \frac{1}{t+1}\, dt = \frac{1}{2}\ln|t-1| - \frac{1}{2}\ln|t+1| + C$$

d) $\int \frac{dx}{1+2\cos^2 x} = \cancel{\int \frac{\cos x\, dx}{\cos x + 2\cos^3 x}} = \int \frac{1}{1+2\frac{1}{t^2+1}} \frac{dt}{t^2+1} =$

$\cancel{\left\{ \begin{array}{l} t = \sin x \\ dt = \cos x\, dx \end{array} \right\}}$ ~~nie działa ... (niepotrzebne potęgi $\cos x$)~~

$= \int \frac{dt}{t^2+1+2} = \frac{1}{3}\int \frac{dt}{\frac{t^2}{3}+1} =$

$\left| \begin{array}{l} t = \operatorname{tg} x \\ dt = (\operatorname{tg}^2 x + 1)\, dx \\ \frac{dt}{t^2+1} = dx \end{array} \right.$

$\cos^2 x = ?$

$t^2 = \operatorname{tg}^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\cos^2 x}{\cos^2 x}$

$t^2 \cos^2 x = 1 - \cos^2 x$

$(t^2+1)\cos^2 x = 1$

$\boxed{\cos^2 x = \frac{1}{t^2+1}}$

$\sin^2 x = 1 - \cos^2 x = \frac{t^2}{t^2+1}$

$= \frac{1}{3}\int \frac{dt}{\left(\frac{t}{\sqrt{3}}\right)^2 + 1} = \left| \begin{array}{l} y = \frac{t}{\sqrt{3}} \\ dy = \frac{1}{\sqrt{3}}\, dt \end{array} \right| =$

$dt = \sqrt{3}\, dy$

$= \frac{1}{3}\sqrt{3}\int \frac{dy}{y^2+1} = \frac{\sqrt{3}}{3} \operatorname{arctg} y + C =$

$= \frac{\sqrt{3}}{3} \operatorname{arctg} \frac{t}{\sqrt{3}} + C =$

$= \frac{\sqrt{3}}{3} \operatorname{arctg}\left(\frac{\operatorname{tg} x}{\sqrt{3}}\right) + C$

$\int \cos^3 x\, dx$ $\left\{ \begin{array}{l} t = \operatorname{tg} x \end{array} \right.$ prowadzi do skomplikowanych rachunków

i) $$\int \frac{dx}{3\sin x + 4\cos x + 5} =$$

$$t = tg\frac{x}{2}$$

$$dt = \left(tg^2\frac{x}{2} + 1\right)\cdot\frac{1}{2}\,dx$$

$$\frac{2dt}{t^2+1} = dx$$

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{2\,\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} + 1} = \frac{2t}{t^2+1}$$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$= \int \frac{\frac{2dt}{t^2+1}}{3\,\frac{2t}{t^2+1} + 4\,\frac{1-t^2}{1+t^2} + 5} =$$

$$= \int \frac{2dt}{6t + 4 - 4t^2 + 5 + 5t^2} =$$

$$= \int \frac{2dt}{t^2 + 6t + 9} =$$

$$= \int \frac{2dt}{(t+3)^2} = \int 2(t+3)^{-2}\,dt = 2\,\frac{(t+3)^{-1}}{-1} + C = \frac{-2}{tg\frac{x}{2} + 3} + C$$

5.4

d) $\int_{-1}^{2} x(1+x^3)\,dx$

$F(x)\Big|_a^b = F(b)-F(a)$

$\int_{-1}^{2}(x+x^4)\,dx = \int_{-1}^{2} x\,dx + \int_{-1}^{2} x^4\,dx = \left(\frac{1}{1+1}x^2 + \frac{1}{5}x^5 + C\right)\Big|_{-1}^{2} =$

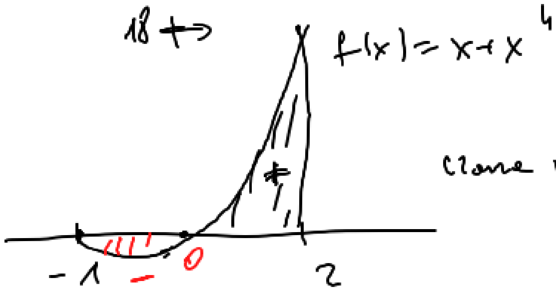
$= \frac{1}{2}x^2 + \frac{1}{5x}^{5} + C$

$\frac{1}{2}2^2 + \frac{1}{5}2^5 - \left(\frac{1}{2}(-1)^2 + \frac{1}{5}(-1)^5\right)$

$\int_{-1}^{2}\ldots = \frac{1}{2}\cdot 4 + \frac{1}{5}\cdot 32 - \frac{1}{2}\cdot 1 + \frac{1}{5}\cdot(-1) = 3 + \frac{32}{5} - \frac{1}{2} - \frac{1}{5} = \frac{5}{2} + \frac{31}{5} =$

$= \frac{25+62}{10} = 8{,}7$

$\int_{-1}^{2}\ldots = \left(\frac{1}{2}4 + \frac{1}{5}\cdot 32 + \cancel{C}\right) - \left(\frac{1}{2} + \frac{1}{5}(-1) + \cancel{C}\right)$



całe pole – ujemne pole

$= 8{,}7$

5b

a)

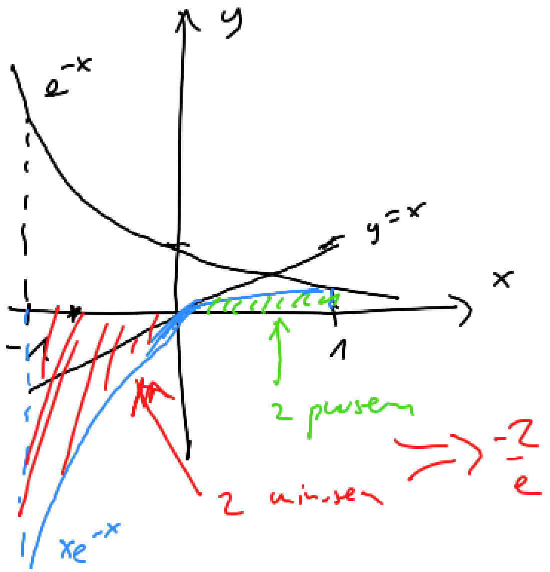
$$\int_{-1}^{1} x\underbrace{e^{-x}}_{= (-e^{-x})'}\,dx = x(-e^{-x})\Big|_{-1}^{1} - \int_{-1}^{1} (-e^{-x})\,dx =$$

$$\boxed{\int_a^b f'g = fg\Big|_a^b - \int_a^b fg'}$$

$$\boxed{\int f'g = fg - \int fg'}$$

$$= 1\cdot(-e^{-1}) - ((-1)\cdot(-e^{+1})) + (-e^{-x})\Big|_{-1}^{1} = -(e^{-1}) - e + (-e^{-1} + e) =$$

$$= -e^{-1} - e - e^{-1} + e = -2\frac{1}{e}$$



5)

$$I = \int_0^{\pi} \underbrace{\sin x}\, e^{\cos x}\, \underbrace{dx} =$$

$t = \cos x$

$dt = -\sin x \, dx$

$-dt = \sin x \, dx$

x	0	π
$t=\cos x$	$\cos 0 = 1$	$\cos \pi = -1$

$$= -\int_1^{-1} e^t \, dt = \int_{-1}^{1} e^t \, dt$$

$$= -e^t \Big|_1^{-1} = -e^{-1} - (-e^1) = e - \frac{1}{e}$$

$$\int \sin x \, e^{\cos x} \, dx = \ldots = -e^{\cos x} + C$$

$$I = -e^{\cos x} \Big|_0^{\pi} = -e^{\cos \pi} + e^{\cos 0} = -e^{-1} + e$$

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$y = 2x - x^2$, $\cancel{x} + y = 0$

$y = -x$

$-x = 2x - x^2$

$0 = 3x - x^2 = x(3 - x)$

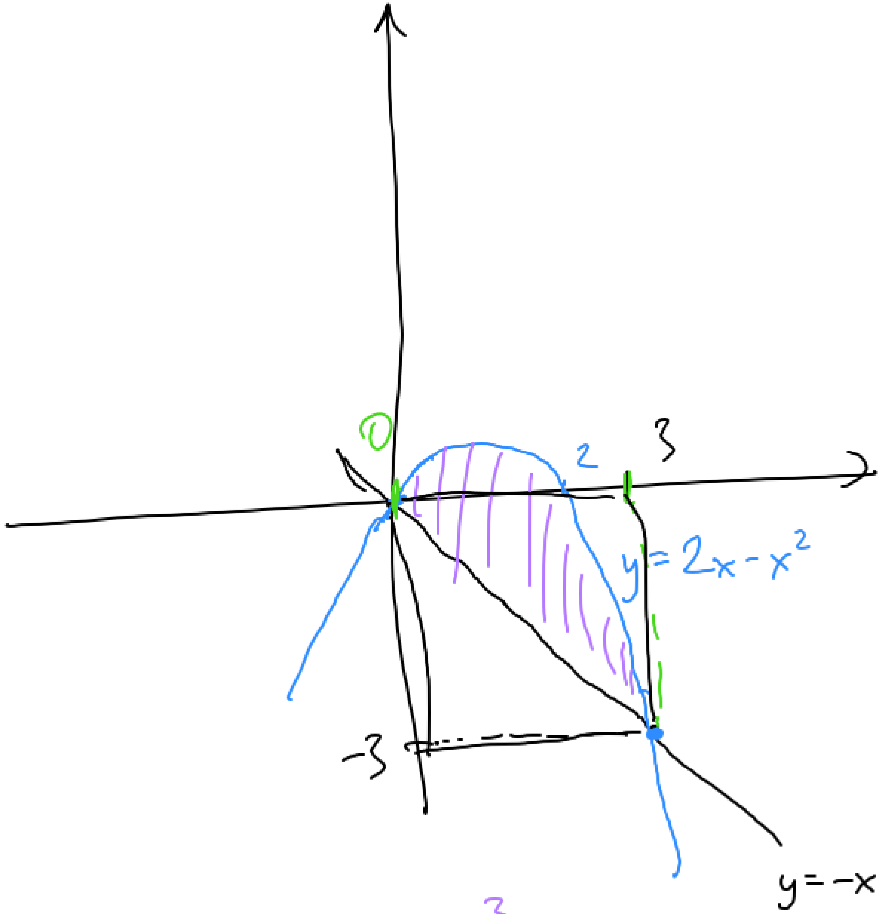
$x = 0 \quad \vee \quad x = 3$

$y = 0 \qquad y = -3$

$$\text{pole} = \int_0^3 (2x - x^2 + x)\,dx =$$

$$= \int_0^3 (3x - x^2)\,dx = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 =$$

$$= \frac{27}{2} - \frac{27}{3} - 0 = \frac{27}{6} = \frac{9}{2} = 4{,}5$$



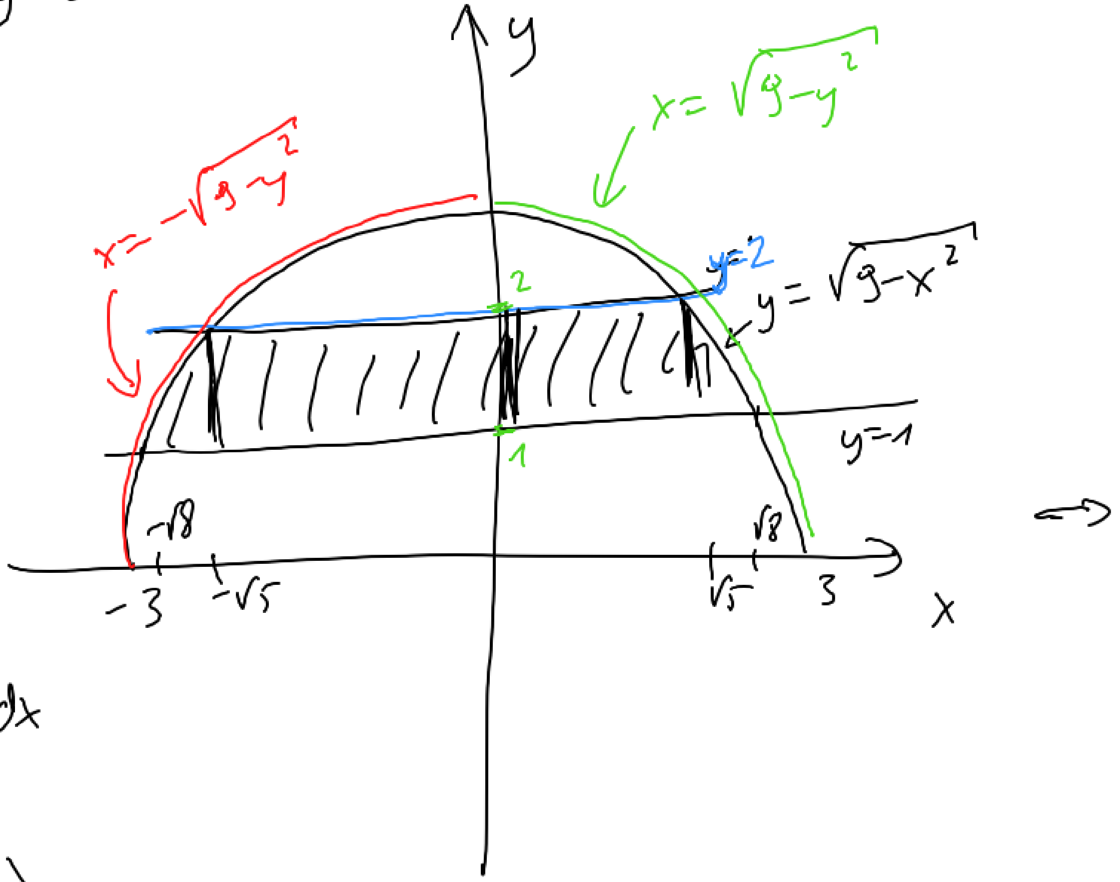
$$P = \int_0^3 \left(2x - x^2 - (-x)\right)dx$$

m) $y=\sqrt{9-x^2}$ $\quad y=1 \quad y=2$

$y^2 = 9 - x^2$

$x^2+y^2=3^2 \rightarrow x^2 = 9-y^2$

$x = \pm\sqrt{9-y^2}$



$$P = \int_{-\sqrt{8}}^{-\sqrt{5}} \left(\sqrt{9-x^2} - 1\right) dx +$$

$$+ \int_{-\sqrt{5}}^{\sqrt{5}} (2-1)\, dx + \int_{\sqrt{5}}^{\sqrt{8}} \left(\sqrt{9-x^2} - 1\right) dx$$

$$= \int_1^2 \left(\sqrt{9-y^2} - \left(-\sqrt{9-y^2}\right)\right) dy$$