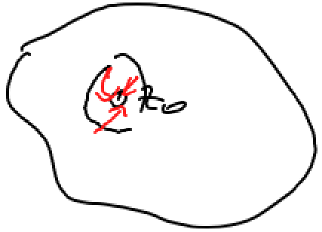
Pochodna zespolona

Niech $E \subset \mathbb{C}$, $f: E \to \mathbb{C}$, $z_0 \in \operatorname{Int} E$.



Def. Jeśli istnieje granica

$$A = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}$$

to mówimy, że f ma pochodną (zespoloną) w punkcie z_0 równą A, piszemy $f'(z_0) = A$.

Powyżej bierzemy granicę po $z \to z_0$, $z \in E \setminus \{z_0\}$.

Př: $\cdot$ $f(z) = z^n \quad (n \in \mathbb{N}) \quad z_0 \in \mathbb{C}$

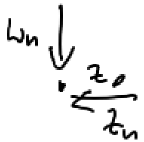
$$f'(z_0) = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} \frac{\left(z^{n-1} \cdot z_0^0 + z^{n-2} z_0^1 + \dots + z_0^{n-1}\right)\cancel{(z - z_0)}}{\cancel{z - z_0}} = n z_0^{n-1}$$

$\cdot$ $f(z) = \frac{1}{z}$, kde $z_0 \neq 0$:

$$f'(z_0) = \lim_{z \to z_0} \frac{\frac{1}{z} - \frac{1}{z_0}}{z - z_0} = \lim_{z \to z_0} \frac{z_0 - z}{z z_0 (z - z_0)} = -\lim_{z \to z_0} \frac{1}{z z_0} = -\frac{1}{z_0^2}$$

$$f(z) = \overline{z}, \qquad f'(z_0) = \lim_{z \to z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0}$$

Pokażemy, że ta granica nie istnieje. Istotnie, dla ciągów:



- $z_n = z_0 + \frac{1}{n} \to z_0$, $\displaystyle \lim_{n\to\infty} \frac{\overline{z_n} - \overline{z_0}}{z_n - z_0} = \lim_{n\to\infty} \frac{\overline{\cancel{z_0} + \frac{1}{n}} - \cancel{\overline{z_0}}}{\cancel{z_0} + \frac{1}{n} - \cancel{z_0}} = \lim \frac{\frac{1}{n}}{\frac{1}{n}} = 1$

- $w_n = z_0 + \frac{i}{n} \to z_0$

$$\lim_{n\to\infty} \frac{\overline{w_n} - \overline{z_0}}{w_n - z_0} = \lim \frac{\cancel{\overline{z_0}} + \overline{\frac{i}{n}} - \cancel{\overline{z_0}}}{\cancel{z_0} + \frac{i}{n} - \cancel{z_0}} = -1$$