1B. Markov processes, heat kernels, Green functions and harmonic measures: hitchhiker's guide to definitions, results and connections (Boundary Harnack principle and friends)

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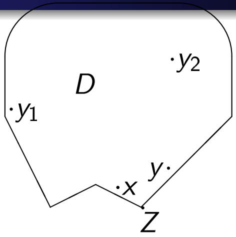
Bounded Lipschitz domain in $\mathbb{R}^d$:



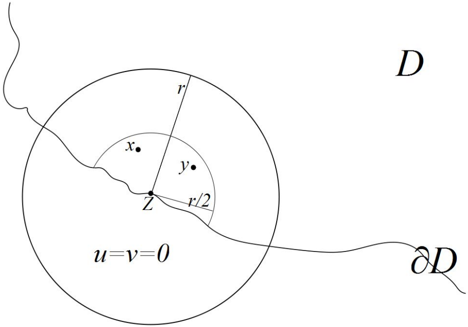
Laplacian: $\Delta = \sum_{i=1}^d \partial_i^2$.
Green function: $G_D(x,y)$.
Harmonic measure: $\omega_D^x(dy)$. For $u \in C^2(\mathbb{R}^d)$ we have:

$$u(x) = \int_{D^c} u(y)\omega_D^x(dy) - \int_D G_D(x,y)\Delta u(y)\,dy.$$

Boundary Harnack inequality (BHI): Ancona; Dahlberg; Wu 1978; Jerison, Kenig 1982; Bass, Burdzy 1988; Aikawa 1985, 2001:

$$\frac{G_D(x,y_1)}{G_D(x,y_2)} \approx \frac{G_D(y,y_1)}{G_D(y,y_2)} \quad \text{if } x,y \text{ are close to } \partial D, \text{ but not to } y_1, y_2.$$

Boundary Harnack inequality, if D is a Lipschitz domain



If $u, v \geq 0$ are harmonic in D and continuous at ∂D near Z, then

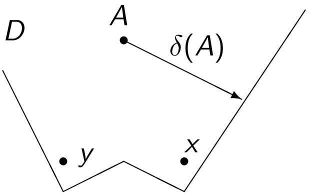
$$M_2^{-1}\frac{u(x)}{v(x)} \leq \frac{u(x)}{v(x)} \leq M_2\frac{u(y)}{v(y)}, \qquad x, y \in D \cap B(Z, r/2)\,.$$

Sharp estimates of the Green function of Lipschitz D for Δ

Let $\phi(x) = G_D(x, x_0) \wedge 1$, where $x_0 \in D$ is fixed and D bounded.

(K. Bogdan, 2000)

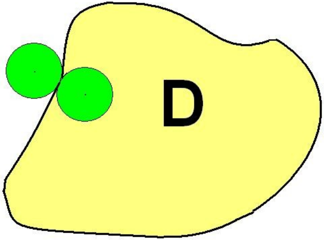
If $d > 2$, then $$G_D(x,y) \approx |x-y|^{2-d} \frac{\phi(x)\phi(y)}{\phi^2(A)}.$$



Here $A = A_{x,y}$ is such that $\delta(A) \approx |x-y| \vee \delta(x) \vee \delta(y)$.

Example: bounded $C^{1,1}$ domains in dimension ≥ 3

Open D is of class $C^{1,1}$ at scale $r > 0$ if for every $Q \in \partial D$ there exist balls $B(x', r) \subset D$ and $B(x'', r) \subset D^c$ tangent at Q.



For such D, $\phi(x) \approx \delta(x)$, $\delta(A_{x,y}) \approx |x-y| \vee \delta(x) \vee \delta(y)$,

$$\begin{aligned} G_D(x,y) &\approx |x-y|^{2-d} \frac{\delta(x)\delta(y)}{\left(\delta(x) \vee \delta(y) \vee |x-y|\right)^2} \\ &\approx |y-x|^{2-d} \left(\frac{\delta(x)\delta(y)}{|x-y|^2} \wedge 1 \right). \end{aligned}$$

The fractional Laplacian

- Recall $d = 1, 2, \ldots$, $0 < \alpha < 2$, and

$$p_t(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^\alpha} d\xi \,, \quad x \in \mathbb{R}^d \,,\ t > 0 \,.$$

  In particular, $p_s * p_t = p_{s+t}$.
- Denote $p(t, x, y) := p_t(y - x)$ and $\nu(dy) := c|y|^{-d-\alpha} dy$.
- Semigroup: $P_t f(x) = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy = p_t * f(x)$.
- Generator: for $f \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{P_t f(x) - f(x)}{t} &= \lim_{\varepsilon\to 0+} \int\limits_{|y|>\varepsilon} [f(x+y) - f(x)]\nu(dy) \\ &=: \Delta^{\alpha/2} f(x), \text{ also denoted } -(-\Delta)^{\alpha/2} f(x). \end{aligned}$$

We write $f(x) \stackrel{c}{\approx} g(x)$, if $c^{-1}g(x) \leq f(x) \leq cg(x)$,

We have

$$p_1(x) \stackrel{c}{\approx} 1 \wedge |x|^{-d-\alpha} \ .$$

We note scaling:

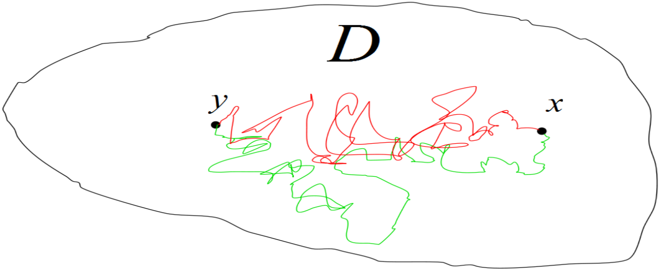
$$p_t(x) = t^{-d/\alpha} p_1(t^{-1/\alpha}x) \stackrel{c}{\approx} t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}} \,, \quad x \in \mathbb{R}^d \,, \ t > 0.$$

Here $c = c(d, \alpha)$, $0 < \alpha < 2$. Hence, $p_t(x) \approx p_{2t}(x)$, etc.

Dirichlet heat kernel

Recall $\tau_D = \inf\{t \geq 0\colon X_t \notin D\}$,

$$p_D(t,x,y) := p(t,x,y) - \mathbb{E}_x\left[\tau_D < t;\ p(t-\tau_D, X_{\tau_D}, y)\right].$$



Green function: $G_D(x,y) := \int_0^\infty p_D(t,x,y)\,dt.$

Recall, $\int_{\mathbb{R}^d} G_D(x,z)\Delta^{\alpha/2}\varphi(z)dz = -\varphi(x)$ for $\varphi \in C_c^\infty(D)$.

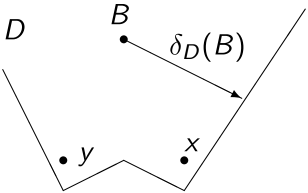
K.B. 1997; Song, Wu 1999; K.B., T.Kulczycki, M.Kwaśnicki 2007: *common* decay rate of nonnegative α-harmonic functions at ∂D:

$G_D(x, x_1) \sim G_D(x, x_2)$ for x close to ∂D. Any open D.

Denote $\delta_D(x) = \mathrm{dist}(x, D^c)$. Let $\phi(x) = G_D(x, x_0) \wedge 1$.

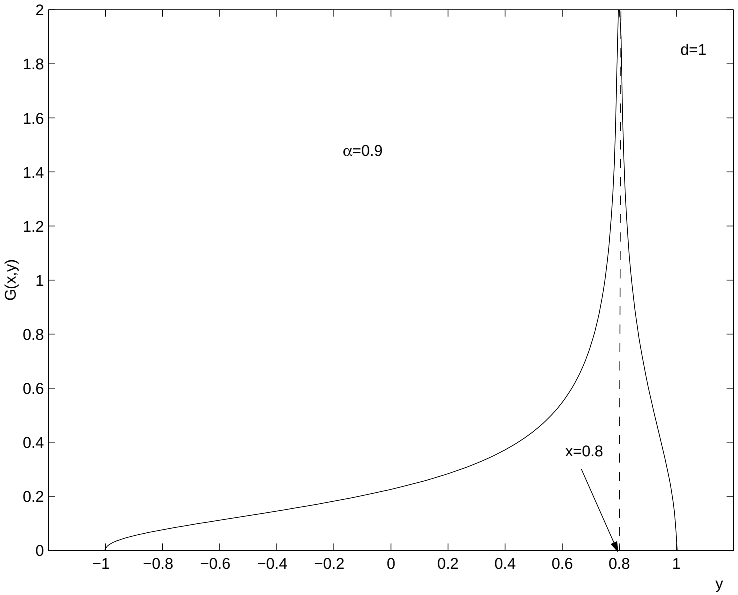
T. Jakubowski 2002: For Lipschitz $D \in \mathbb{R}^d$ and $d > \alpha$,

$$G_D(x, y) \approx |x - y|^{\alpha - d} \, \phi(x)\phi(y)/\phi^2(B) \, .$$

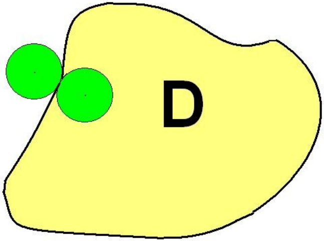


Here $\delta_D(B) \approx |x - y| \vee \delta_D(x) \vee \delta_D(y)$, as before.

Example: Green function $y \mapsto G_{(-1,1)}(x, y)$ of the interval



Example: bounded $C^{1,1}$ domains, $0 < \alpha < 2$



$$\phi(x) \approx (\,|y - x| \vee \delta(x) \vee \delta(y)\,)^{\alpha/2}.$$

(Kulczycki 1997, Chen, Song 1998)

If D is bounded and $C^{1,1}$, then

$$G_D(x, y) \approx |x - y|^{\alpha - d} \left(\frac{\delta_D(x)\delta_D(y)}{|x - y|^2} \wedge 1 \right)^{\alpha/2} .$$

Not hitting the boundary upon exit and the Poisson kernel

We have $\omega_D^x(\partial D) = 0$ if $x \in D$ and D is Lipschitz.

In fact, it is absolutely continuous on D^c, with density $P_D(x, y)$.

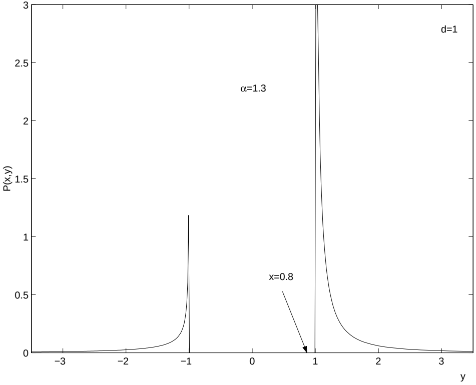
Here P_D, the *Poisson kernel* of D, is defined by

$$P_D(x, y) := \int_D G_D(x, v)\nu(v, y)\, dv\,, \quad x \in \mathbb{R}^d\,,\ y \in D^c\,.$$

Poisson kernel of the ball

Let $B_r = \{x \in \mathbb{R}^d \; : \; |x| < r\}$, $\mathcal{C}_{d,\alpha} = \Gamma(d/2)\pi^{-1-d/2}\sin(\pi\alpha/2)$.
By a calculation of M. Riesz (R. Blumenthal, R. Getoor, D. Ray),

$$P_{B_r}(x,y) = \mathcal{C}_{d,\alpha}\left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^{\alpha/2}\frac{1}{|x-y|^d}\,, \quad |x| < r\,,\, |y| > r\,.$$



Approximate factorization of Poisson kernel

Recall, for $B_r = \{x \in \mathbb{R}^d \,:\, |x| < r\}$,

$$P_{B_r}(x,y) = \mathcal{C}_{d,\alpha}\left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^{\alpha/2}\frac{1}{|x-y|^d}\,, \quad |x|<r\,,\,|y|>r\,.$$

This yields, e.g., the relative constancy/Harnack inequality in x. Moreover, if x and y are not close to each other,

$$P_{B_r}(x,y) \approx (r^2-|x|^2)^{\alpha/2}\cdot(|y|^2-r^2)^{-\alpha/2}|y|^{-d}.$$

Theorem (K. B. , T. Kulczycki, M. Kwaśnicki, 2007)

There is $C_{d,\alpha}$, depending only on d and α, such that

$$P_D(x_1,y_1)P_D(x_2,y_2) \le C_{d,\alpha}\, P_D(x_1,y_2)P_D(x_2,y_1)\,,$$

whenever $r>0$, $x_1,x_2 \in D\cap B_{r/2}$ and $y_1,y_2 \in D^c \cap B_r^c$.

Harmonic functions

Definition

$u \geq 0$ is α-harmonic in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = \mathbb{E}^x u(X(\tau_U)) < \infty, \quad x \in U,$$

for every bounded open set U satisfying $\overline{U} \subset D$.

- Or, $\Delta^{\alpha/2} u = 0$ (distr.) on D. (K. B. , T. Byczkowski, 1999)
- $G_D(\cdot, y)$ and $G_D(y, \cdot)$ are α-harmonic in $D \setminus \{y\}$...
- ... so are $u_1(x) := \int_{D^c} f(z)\omega_D^x(dz)$,
  $u_2(x) := \int_{D^c} f(y)P_D(x, y)dy$, and
  $u_2(x) := \int_{D^c} P_D(x, y)\lambda(dy)$.
- Problem: give (Martin) representation of (globally) nonegative functions u which are α-harmonic on D.

Points accessible from D

We will say that $y \in \mathbb{R}^d$ is *accessible* (from D) if

$$P_D(x_0, y) = \int_{\mathbb{R}^d} G_D(x_0, v)\nu(v, y)dv = \infty .$$

The condition is independent of the choice of $x_0 \in D$, and means that D is "large/thick" around y. We say ∞ is accessible if

$$\mathbb{E}_x \tau_D := s_D(x) := \int_{\mathbb{R}^d} G_D(x_0, v)dv = \infty .$$

We define *Martin boundary*:

$$\partial_M D. := \{y \in \partial D \cup \{\infty\} \ : \ \ y \text{ is accessible}\} .$$

Limits of ratios of Poisson integrals

For function $q > 0$ on a set $U \neq \emptyset$ we define *relative oscillation:*

$$\rho_U q \; = \sup_{x \in U} q(x) / \inf_{x \in U} q(x) \,.$$

Lemma

For every $\eta > 0$ there exists $r > 0$ such that

$$\rho_{D \cap B_r} \frac{P_D[\lambda_1]}{P_D[\lambda_2]} \leq 1 + \eta$$

for open $D \subset B_1$ and $\lambda_1, \lambda_2 \geq 0$ on B_1^c with finite $P_D[\lambda_i]$.

A consequence: $\lim \frac{P_D[\lambda_1](x)}{P_D[\lambda_2](x)}$ exists as $D \ni x \to y \in \partial D$.

Martin kernel of (Greenian) D, $\Delta^{\alpha/2}$ and $0 < \alpha < 2$

We fix (a reference point) $x_0 \in D$ and define

$$M_D(x,y) = \lim_{D \ni v \to y} \frac{G_D(x,v)}{G_D(x_0,v)}, \quad x \in \mathbb{R}^d, \; y \in \partial_* D.$$

Note that $M_D(x,y) = 0$ for (most) $x \in D^c$.

Theorem

The (unique) limit exists for every limiting point of D. $M_D(x,y)$ is α-harmonic in x on D **if and only if** *y is accessible.*

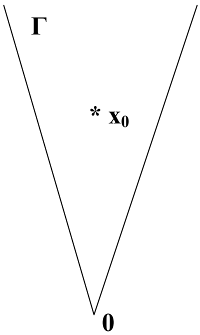
Explanation:

$$G_D(x,v) = P_{D \cap B_\rho}[G_D(x,u)du](v),$$

if $v \in D \cap B_\rho$, $x, x_0 \notin D \cap B_\rho \ni 0$, and

$$G_D(x_0,v) = P_{D \cap B_\rho}[G_D(x_0,u)du](v).$$

Γ is a cone.
Let $M(x) := M_\Gamma(x, \infty)$:
If $m(x)$ is another candidate,
let $C = \inf_\Gamma \frac{M(x)}{m(x)}$.
By BHP, $0 < C < \infty$.
We have $M \geq Cm$ or
$M - Cm =: R \geq 0$.
If $R > 0$ then $R \geq \varepsilon m$,
thus $M \geq (C + \varepsilon)m$. !?!
Existence of limits: related.
In general, subtract w/caution
because of non-locality.



A word on the point at infinity

Consider the inversion with respect to the unit sphere in $\mathbb{R}^d$:

$$Tx = \frac{x}{|x|^2}, \quad x \neq 0 .$$

Inversion reduces potential theoretic problems at ∞ to those at 0. Let $TD = \{Tx : x \in D\}$. We have

$$G_D(x,v) = |x|^{\alpha-d}|v|^{\alpha-d} G_{TD}(Tx, Tv), \quad x\,.v \neq 0 ,$$

Or, $G_D(x,v) = K_x K_v G_{TD}(x,v)$, where $Kf(x) = |x|^{\alpha-d} f(x/|x|^2)$. Similarly for M_D.

Structure of nonnegative α-harmonic functions on D

Finite Poisson integral of $\lambda \geq 0$ is α-harmonic w/outer charge λ:

$$P_D[\lambda](x) := \int_{D^c} P_D(x, y)\lambda(dy)\,, \quad x \in D\,.$$

If measure $\mu \geq 0$ is finite on $\partial_M D$, then so is

$$M_D[\mu](x) := \int_{\partial_M D} M_D(x, y)\mu(dy)\,, \quad x \in \mathbb{R}^d\,.$$

Theorem

Let D be Greenian. For every function $f \geq 0$ on D, α-harmonic in D with outer charge $\lambda \geq 0$, there is a **unique** *$\mu \geq 0$ on $\partial_M D$,*

$$f(x) = P_D[\lambda](x) + M_D[\mu](x)\,, \quad x \in D\,.$$

For non-Greenian D, f must be constant on D.

References: K. Bogdan, R. Bañuelos; K. Bogdan, T. Kulczycki, M. Kwaśnicki; K. Michalik, Biočić.

Estimates of the heat kernel

$$p_t(x) \approx t^{-d/\alpha} \wedge \frac{t}{|x|^{d+\alpha}}$$

Upper heat kernel bounds for convex domains: B. Siudeja 2006.

(Z.-Q. Chen, P. Kim, R. Song 2008)

If D is $C^{1,1}$ then for $0 < t \le 1$, $x, y \in \mathbb{R}^d$,

$$p_D(t,x,y) \approx \left(\frac{\delta_D(x)}{t^{1/\alpha}} \wedge 1 \right)^{\alpha/2} p(t,x,y) \left(\frac{\delta_D(y)}{t^{1/\alpha}} \wedge 1 \right)^{\alpha/2} .$$

(K.B., T. Grzywny, M. Ryznar 2011)

If D is κ-fat, then for $x, y \in \mathbb{R}^d$, $0 < t \le 1$,

$$p_D(t,x,y) \approx \mathbb{P}^x(\tau_D > t)\, p(t,x,y)\, \mathbb{P}^y(\tau_D > t)\,,$$

and $\mathbb{P}^x(\tau_D > t)$ has sharp semi-explicit bounds.

Time-line for kernel estimates for Δ

- Sharp explicit estimates of the Green function $C^{1,1}$ domains: Z. Zhao 1986.
- Sharp estimates for Lipschitz domains: K.B. 2000.
- Explicit qualitatively sharp estimates for the heat kernel for $C^{1,1}$ domains: Q.-S. Zhang 2002.
- Qualitatively sharp estimates for the heat kernel for Lipschitz domains: N. Varopulous 2003.

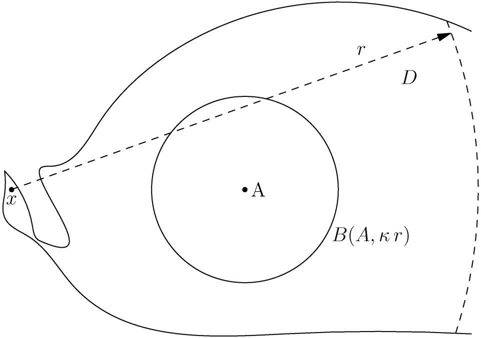
Time-line of kernel estimates for $\Delta^{\alpha/2}$

- Green function and $C^{1,1}$ open sets: T. Kulczycki 1997, Z.-Q. Chen, R. Song 1998.
- Green function and open Lipschitz sets: T. Jakubowski 2002.
- Upper bounds for the heat kernel of convex domains: B. Siudeja 2006.
- Heat kernel for $C^{1,1}$ open sets: P. Kim, R. Song, Z. Chen 2008:
$$p_D(t,x,y) \approx \left(1 \wedge \frac{\delta^{\alpha/2}(x)}{\sqrt{t}}\right) p(t,x,y) \left(1 \wedge \frac{\delta^{\alpha/2}(y)}{\sqrt{t}}\right) .$$
  Here $\delta(x) = \mathrm{dist}(x, D^c)$ oraz $0 < t \leq 1\,,\ x, y \in \mathbb{R}^d$.
- κ-fat sets: K.B., T. Grzywny, M. Ryznar 2008-2011.
- P. Kim, R. Song, Z. Chen, Z. Vondraček, T. Kumagai, A. Grigorya, K. Bogdan, T. Grzywny, M. Ryznar, V. Knopova, A. Kulik, R. Schilling, T. Jakubowski, K. Szczypkowski: further subordinated Brownian motions, metric spaces, perturbations of semigroups, parametrix ...

κ-fat sets:

D is (κ, r)-fat at x, if there is a ball $\mathrm{B}(\mathrm{A}, \kappa \mathrm{r}) \subset \mathrm{D} \cap \mathrm{B}(\mathrm{x}, \mathrm{r})$.

Denote $A_r(x) := A$. We have: $\delta(A_r) \approx r \vee \delta(x)$.



K.B., T. Grzywny, M. Ryznar 2008-2011

If D is κ-fat and $x, y \in \mathbb{R}^d$, then

$$p_D(t,x,y) \approx \mathbb{P}_x(\tau_D > t)\, p(t,x,y)\, \mathbb{P}_y(\tau_D > t)\,, \quad 0 < t \leq 1.$$

- Motivation: let $c(t) = p(t,0,0) \geq \sup_{z,y\in\mathbb{R}^d} p_D(t,z,y)$,

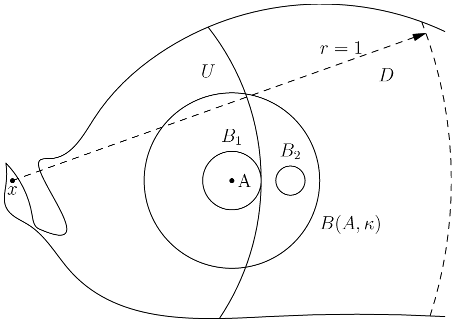
$$\begin{aligned} p_D(3t,x,y) &= \int\int p_D(t,x,z)p_D(t,z,w)p_D(t,w,y)dwdz \\ &\leq \mathbb{P}_x(\tau_D > t)c(t)\mathbb{P}_y(\tau_D > t)\,. \end{aligned}$$

- The proof of the off-diagonal estimates uses BHP, the Lévy system of X and comparability of p and ν at infinity.

Where BHP is used:

If D is $(\kappa, 1)$-fat at $x \in D$, then

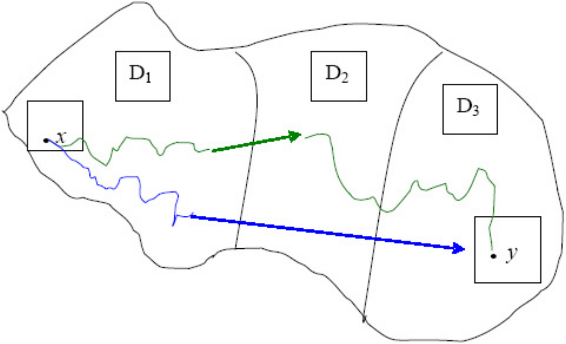
$$\mathbb{P}_x(\tau_D > 1/3) \approx \mathbb{P}_x(\tau_D > 1) \approx \mathbb{P}_x(\tau_D > 3) \approx \mathbb{P}_x(X_{\tau_U} \in D) \approx E^x \tau_U.$$



The idea for the upper bound:

Let $D_1, D_3 \subset D$, $\operatorname{dist}(D_1, D_3) > 0$ *and* $D_2 := D \setminus (D_1 \cup D_3)$.
If $x \in D_1$ *and* $y \in D_3$, *then*

$$p_D(1,x,y) \le \mathbb{P}_x(X_{\tau_{D_1}} \in D_2) \cdot \sup_{s<1,\, z\in D_2} p(s,z,y) + E^x \tau_{D_1} \cdot \sup_{u\in D_1,\, z\in D_3} \nu(z-u).$$



Estimates of survival probability

Let $s_D(x) := E_x\tau_D = \int G_D(x,y)dv$, if finite, else let $s_D(x) := M_D(x,\infty) = \lim_{D\ni y\to\infty} G_D(x,y)/G_D(x_0,y)$.

By homogeneity,

$$\frac{s_{rD}(rx)}{s_{rD}(ry)} = \frac{s_D(x)}{s_D(y)}\,, \quad x,y\in D,\ r>0\,.$$

If D is $(\kappa,t^{1/\alpha})$-fat at x and y, then

$$\mathbb{P}_x(\tau_D > t) \stackrel{C}{\approx} \frac{s_D(x)}{s_D(A_{t^{1/\alpha}}(x))}\,,$$

where $C = C(d,\alpha,\kappa)$. Therefore

$$p_D(t,x,y) \stackrel{C^2}{\approx} \frac{s_D(x)}{s_D(A_{t^{1/\alpha}}(x))} p(t,x,y) \frac{s_D(y)}{s_D(A_{t^{1/\alpha}}(y))}\,.$$

Application to the ball

For $R > 0$ and $D = B(0,R) \subset \mathbb{R}^d$, we have $s_D(x) \approx \delta^{\alpha/2}(x) R^{\alpha/2}$.

for $t \le R^\alpha$, $s_D(A_{t^{1/\alpha}}(x)) \stackrel{C}{\approx} (t^{1/\alpha} \vee \delta(x))^{\alpha/2} R^{\alpha/2}$, thus

$$\mathbb{P}_x(\tau_D > t) \stackrel{C}{\approx} \frac{\delta^{\alpha/2}(x)}{(t^{1/\alpha} \vee \delta(x))^{\alpha/2}} = \left(1 \wedge \frac{\delta(x)}{t^{1/\alpha}}\right)^{\alpha/2}, \quad x \in D,$$

and

$$p_D(t,x,y) \stackrel{C}{\approx} \left(1 \wedge \frac{\delta^{\alpha/2}(x)}{t^{1/2}}\right) p(t,x,y) \left(1 \wedge \frac{\delta^{\alpha/2}(y)}{t^{1/2}}\right), \quad x, y \in \mathbb{R}^d.$$

Here $C = C(d, \alpha)$ and $t \le R^\alpha$.

Similarly for all bounded $C^{1,1}$ open sets.

Application to the Cauchy process on $D = (-1, 1)^c$

Here $\alpha = d = 1$, $s_D(x) \approx \log(1 + \delta^{1/2}(x))$. We have

$$\mathbb{P}_x(\tau_D > t) \approx \frac{\log(1 + \delta^{1/2}(x))}{\log(1 + (t \vee \delta(x))^{1/2})} = 1 \wedge \frac{\log(1 + \delta^{1/2}(x))}{\log(1 + t^{1/2})}.$$

Thus,

$$p_D(t, x, y) \approx \left(1 \wedge \frac{\log(1 + \delta^{1/2}(x))}{\log(1 + t^{1/2})}\right) p(t, x, y) \left(1 \wedge \frac{\log(1 + \delta^{1/2}(y))}{\log(1 + t^{1/2})}\right).$$

Here $t > 0$ and $x, y \in \mathbb{R}^d$ are arbitrary.

Lipschitz cones with homogeneity exponent $\beta \in (0, \alpha)$

For cone Γ and arbitrary $x, y \in \mathbb{R}^d$, $0 < t < \infty$,

$$\frac{p_\Gamma(t,x,y)}{p(t,x,y)} \approx \frac{\left(1 \wedge \frac{\delta(x)}{t^{1/\alpha}}\right)^{\alpha/2}}{\left(1 \wedge \frac{|x|}{t^{1/\alpha}}\right)^{\alpha/2-\beta}} \frac{\left(1 \wedge \frac{\delta(y)}{t^{1/\alpha}}\right)^{\alpha/2}}{\left(1 \wedge \frac{|y|}{t^{1/\alpha}}\right)^{\alpha/2-\beta}}.$$

Further collaborations

- Byczkowski, 1999-2001: Feynman-Kac semigroups
- Dziubański, Jakubowski, Pilarczyk, Sydor, Szczypkowski, 2007-present: additive perturbations (of generators of) Markovian semigroups
- Michalik, Ryznar, Dyda, Luks: relative Fatou theorem and Hardy spaces
- Grzywny, Ryznar, 2014-15: isotropic Lévy semigroups
- Sztonyk, Knopova, 2005-present: estimates for anisotropic nonlocal operators
- Dyda, Kim, Merz, 2004-present: Hardy-type (in)equalities for non-local Dirichlet forms
- Kwaśnicki, Kumagai, 2015: BHP
- Pietruska-Pałuba, Rutkowski, Lenczewska, Grzywny, Jakubowski, 2022-present: the L^p setting
- Pilarczyk, Leżaj, Knosalla, Grzywny, Kim, Palmowski, Armstrong, 2018-present: Yaglom limits
- Hansen, Kania, Jarohs, 2020-present: semilinear equations

Open problems

- Homogeneity exponent (of the Martin kernel with the pole at infinity) for cones
- Limits of $p_D(t,x,y)/p_D(t,x,y)$ as $y \to z \in \partial D$
- Construction of Markovian semigroups with prescribed system of jumps
- Other boundary conditions and applications
- The L^p setting