4. Sobolev-Bregman forms in elliptic and parabolic problems

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Plan: 1. Bregman divergence 2. Hardy inequality in L^p
(3. Hardy-Stein and Douglas formulas; optional)

[14] *Optimal Hardy inequality for the fractional Laplacian on* L^p, 2022,
KB, T. Jakubowski, J. Lenczewska, K. Pietruska-Pałuba

[12] *Nonlinear nonlocal Douglas identity*, 2023
KB, T. Grzywny, K. Pietruska-Pałuba, A. Rutkowski

Classical Hardy inequalities

For historical account see Kufner, Maligranda, Persson [30]. Hardy [24] initiated the subject in 1920 by proving that

$$\int_0^\infty \left[u'(x)\right]^2 dx \geq \frac{1}{4}\int_0^\infty \frac{u(x)^2}{x^2} dx,$$

for absolutely continuous u with $u(0) = 0$ and $u' \in L^2(0,\infty)$.
The classical Hardy inequality in $\mathbb{R}^d$ for $d \geq 2$ is

$$\int_{\mathbb{R}^d} |\nabla u(x)|^2 \geq \frac{(d-2)^2}{4}\int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^2} dx.$$

For symmetric Dirichlet form $\mathcal{E}$, Fitzsimmons [21] proposed this:

If $\mathcal{L}$ is the generator of $\mathcal{E}$, $h \geq 0$ and $\mathcal{L}h \leq 0$ (superharmonic), then

$$\mathcal{E}(u,u) \geq \int u^2 \frac{-\mathcal{L}h}{h}.$$

Fractional Laplacian

Once and for all let $d \in \mathbb{N}$ and $\alpha \in (0,2)$. Consider

$$\Delta^{\alpha/2}u(x) := -(-\Delta)^{\alpha/2}u(x) := \lim_{\epsilon\to 0+} \int_{|y-x|>\epsilon} (u(y) - u(x))\, \nu(x-y)\, dy,$$

where $\nu(z) = \mathcal{A}_{d,-\alpha}|z|^{-d-\alpha}$, $z \in \mathbb{R}^d$ (Lévy measure density),

$$\mathcal{A}_{d,-\alpha} = 2^\alpha \Gamma\big((d+\alpha)/2\big)\pi^{-d/2}/|\Gamma(-\alpha/2)|,$$

and, say, $u \in C_c^2(\mathbb{R}^d)$. Let

$$\mathcal{E}[u] := \mathcal{E}(u,u) := \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (u(x)-u(y))^2 \nu(x-y)\, dy\, dx,$$

and $\mathcal{D}(\mathcal{E}) := \{u \in L^2(\mathbb{R}^d) : \mathcal{E}[u] < \infty\}$.

Hardy identity on $L^2(\mathbb{R}^d)$

By [6], if $\alpha < d$, $0 \le \beta \le d-\alpha$, and $u \in L^2(\mathbb{R}^d)$, then

$$\mathcal{E}[u] = \kappa_\beta \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha}\,dx + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left[\frac{u(x)}{h_\beta(x)} - \frac{u(y)}{h_\beta(y)}\right]^2 h_\beta(x) h_\beta(y) \nu(x-y)\,dy\,dx,$$

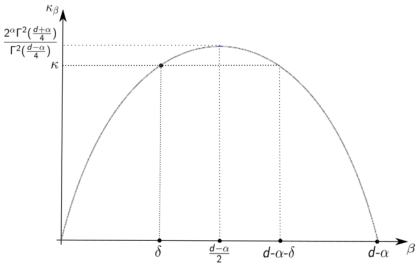
where $h_\beta(x) := |x|^{-\beta}$, and

$$\kappa_\beta = \frac{2^\alpha \Gamma\left(\frac{\beta+\alpha}{2}\right)\Gamma\left(\frac{d-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{d-\beta-\alpha}{2}\right)};$$

see earlier Frank, Lieb and Seiringer [22] for $u \in C_c^\infty(\mathbb{R}^d)$.

Note that $\kappa_\delta = \kappa_{d-\alpha-\delta}$ (symmetry w/r to $\delta = (d-\alpha)/2$).

Figure: The function $\beta \mapsto \kappa_\beta$.



$$\kappa_{(d-\alpha)/2} = 2^\alpha \Gamma\left(\frac{d+\alpha}{4}\right)^2 \Gamma\left(\frac{d-\alpha}{4}\right)^{-2},$$

The following fractional Hardy inequality is optimal in L^2:

$$\mathcal{E}[u] \geq \kappa_{(d-\alpha)/2} \int_{\mathbb{R}^d} \frac{u(x)^2}{|x|^\alpha}\, dx,$$

see Herbst [25], Beckner [5] and Yafaev [40].

The $L^p(\mathbb{R}^d)$ setting: the Sobolev-Bregman form

For $p \in (1, \infty)$ and $u : \mathbb{R}^d \to \mathbb{R}$ we define the p-form,

$$\mathcal{E}_p[u] := \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(u(x)^{\langle p-1 \rangle} - u(y)^{\langle p-1 \rangle}) \nu(x-y) \, dy \, dx.$$

Here and below $a^{\langle k \rangle} := |a|^k \operatorname{sgn} a$. We have (nearly optimal)

$$\frac{4(p-1)}{p^2} (b^{\langle p/2 \rangle} - a^{\langle p/2 \rangle})^2 \le (b-a)(b^{\langle p-1 \rangle} - a^{\langle p-1 \rangle}) \le 2(b^{\langle p/2 \rangle} - a^{\langle p/2 \rangle})^2,$$

see Liskevich, Perelmuter and Semenov [32]. Thus, for $u \in L^p(\mathbb{R}^d)$,

$$\mathcal{E}_p[u] \ge \frac{4(p-1)}{p^2} \mathcal{E}_2[u^{\langle p/2 \rangle}] \ge \frac{4(p-1)}{p^2} \kappa_{(d-\alpha)/2} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} \, dx.$$

The inequality is given, e.g., in Cialdea and Maz'ya [18].

Our goal is, among others, to *improve the constant*.

Bregman divergence

Recall (the French power):

$$x^{<\kappa>} = |x|^\kappa \operatorname{sgn}(x), \quad \kappa, x \in \mathbb{R}.$$

E.g., $x^{\langle 0\rangle} = \mathrm{sgn}(\mathrm{x})$, $\sqrt[3]{x} = x^{\langle 1/3\rangle}$ and $x^{\langle 2\rangle} \neq x^2$ as functions on $\mathbb{R}$.

We have $(|x|^\kappa)' = \kappa x^{<\kappa-1>}$ and $(x^{<\kappa>})' = \kappa|x|^{\kappa-1}$ for $x \neq 0$.

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Recall that $p \in (1, \infty)$. Define (Bregman divergence),

$$F_p(a,b) = |b|^p - |a|^p - p a^{<p-1>}(b-a), \quad a, b \in \mathbb{R}.$$

E.g., $F_2(a,b) = (b-a)^2$ and $F_4(a,b) = (b-a)^2(b^2 + 2ab + 3a^2)$.
Note that $F_p(a,b)$ is the second-order Taylor remainder of $|x|^p$.
It is an example of *Bregman divergence*, see, e.g., Sprung [37].

Estimates and algebra of F_p

Recall that $F_p(a,b) = |b|^p - |a|^p - pa^{<p-1>}(b-a)$. By the convexity of $|x|^p$, we have $F_p \geq 0$. Moreover,

$$F_p(a,b) \approx (b-a)^2(|a|+|b|)^{p-2}, \quad a,b \in \mathbb{R},$$

see Pinchover, Tertikas, Tintarev [35], Bogdan, Dyda, Luks [7] and Bogdan, Więcek [15]. Again, we also have [32]

$$F_p(a,b) \approx (a^{<p/2>} - b^{<p/2>})^2.$$

Note $|b-a|^p \lesssim F_p(a,b)$ if $p \geq 2$, $F_p(a,b) \lesssim |b-a|^p$ if $p \leq 2$. In general $F_p(a,b) \neq F_p(b,a)$, but (the symmetrization yields)

$$\frac{1}{2}(F_p(a,b) + F_p(b,a)) = \frac{p}{2}(b-a)(b^{\langle p-1\rangle} - a^{\langle p-1\rangle}).$$

Thus, $\mathcal{E}_p[u] \approx \mathcal{E}[u^{<p/2>}]$.

Hardy identity and inequality on L^p

Recall $h_\beta(x) = |x|^{-\beta}$, $x, \beta \in \mathbb{R}^d$.

Theorem (1)

If $0 < \alpha < d \wedge 2$, $0 \le \beta \le (d-\alpha) \wedge (d-\alpha)/(p-1)$, $h = h_\beta$ *and* $u \in L^p(\mathbb{R}^d)$, *then*

$$\mathcal{E}_p[u] = \frac{\kappa_{(p-1)\beta} + (p-1)\kappa_\beta}{p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} dx$$
$$+ \frac{1}{p} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_p\left(\frac{u(x)}{h(x)}, \frac{u(y)}{h(y)}\right) h(x)^{p-1} h(y) \nu(x-y)\, dy\, dx.$$

In particular, for $\beta = (d-\alpha)/p$ we obtain

$$\mathcal{E}_p[u] \ge \kappa_{(d-\alpha)/p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} dx, \qquad u \in L^p(\mathbb{R}^d).$$

Optimality

Recall,

$$\mathcal{E}_p[u] \geq \kappa_{(d-\alpha)/p} \int_{\mathbb{R}^d} \frac{|u(x)|^p}{|x|^\alpha} dx, \qquad u \in L^p(\mathbb{R}^d). \qquad (1)$$

It turns out (by calculus) that for $p \neq 2$ we have

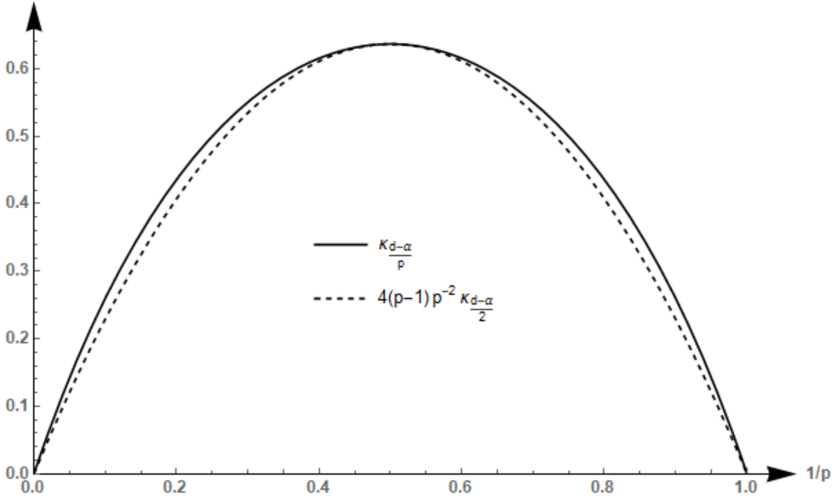
$$\kappa_{(d-\alpha)/p} > \frac{4(p-1)}{p^2} \frac{2^\alpha \Gamma\left(\frac{d+\alpha}{4}\right)^2}{\Gamma\left(\frac{d-\alpha}{4}\right)^2}.$$

Here is a deeper result.

Theorem (2)

The constant in (1) *is sharp.*

Comparison of the constants for $d = 3$, $\alpha = 1$



Results: Applications

Let $\tilde{P}_t$ be the F-K semigroup generated by $\Delta^{\alpha/2} + \kappa_\delta |x|^{-\alpha}$.

Theorem (3)

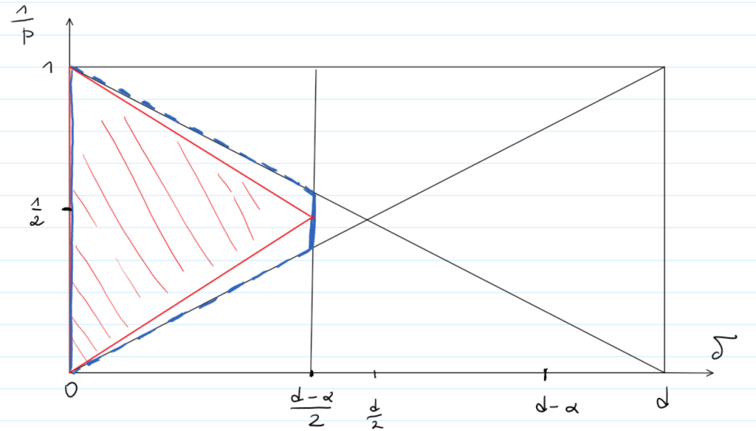
Let $0 < \alpha < d$, $1 < p < \infty$ and $0 < t < \infty$. The operator $\tilde{P}_t$ is a contraction on $L^p(\mathbb{R}^d)$ if and only if $\kappa_\delta \le \kappa_{(d-\alpha)/p}$.

Recall that (for $\alpha = 2$) $\Delta + \kappa|x|^{-2}$ generates a contraction semigroup on $L^p(\mathbb{R}^d)$ iff $\kappa \le \kappa_{(d-2)/p} = (d-2)^2(p-1)p^{-2}$, see Kovalenko, Perelmuter and Semenov [29], Liskevich and Semenov [34] and Arendt, Goldstein and Goldstein [1].

Theorem (4)

Let $1 < p < \infty$ and $0 < t < \infty$. The operator $\tilde{P}_t$ is bounded on $L^p(\mathbb{R}^d)$ if and only if $\delta < d/p^*$, where $p^* = \max\{p, p/(p-1)\}$.

Illustration: The range of admissible p in Theorem (3) is marked in red, and in Theorem (4) – in blue.



Let $p_t(x,y) \sim \Delta^{\alpha/2}$. We have $p_t(x,y) = p_t(x-y)$ and (scaling):

$$p_t(z) = t^{-\frac{d}{\alpha}} p_1(t^{-\frac{1}{\alpha}} z), \quad t > 0, \quad z \in \mathbb{R}^d .$$

It is well known that $p_t(x,y) \approx \min\left(t^{-d/\alpha}, t|x-y|^{-d-\alpha}\right)$, hence

$$p_t(x,y)/t \leq c\nu(x-y), \quad t > 0, \quad x, y \in \mathbb{R}^d.$$

Also, $p_t(x,y)/t \to \nu(x-y)$ as $t \to 0^+$.

For $u \in L^p(\mathbb{R}^d)$, $v \in L^{p/(p-1)}(\mathbb{R}^d)$ and $t > 0$, let

$$\mathcal{E}^{(t)}(u,v) := \frac{1}{t}\langle u - P_t u, v\rangle.$$

Then, for $u \in L^p(\mathbb{R}^d)$, $u \in \mathcal{D}_p(\Delta^{\alpha/2})$ (respectively),

$$\mathcal{E}_p[u] = \lim_{t\to 0} \mathcal{E}^{(t)}(u, u^{\langle p-1\rangle}) \overset{(resp.)}{=} -\langle \Delta^{\alpha/2} u, u^{\langle p-1\rangle}\rangle.$$

The α-stable convolution semigroup

Recall $d \in \{1, 2, \ldots\}$, $0 < \alpha < 2$ and

$$\nu(z) = \mathcal{A}_{d,-\alpha}|z|^{-d-\alpha}, \quad z \in \mathbb{R}^d.$$

In a connection to the Lévy-Khintchine formula,

$$\int_{\mathbb{R}^d} (1 - \cos \xi \cdot x)\, \nu(|x|)\, \mathrm{d}x = |\xi|^\alpha, \quad \xi \in \mathbb{R}^d,$$

and for every $t > 0$ there is a smooth function $p_t > 0$ such that

$$\int_{\mathbb{R}^d} \mathrm{e}^{i\xi \cdot x} p_t(x)\, \mathrm{d}x = \mathrm{e}^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d.$$

Of course, $p_s * p_t = p_{s+t}$. We can also treat p_t by *subordination*:

$$p_t(x) = \int_0^\infty g_s(x)\eta_t(s)\, ds, \quad t > 0, \quad x \in \mathbb{R}^d.$$

For $\alpha < d$ and $\beta \in (0, d)$, we let

$$f_\beta(t) = c t_+^{(d-\alpha-\beta)/\alpha}, \quad t \in \mathbb{R}.$$

Here $c \in (0, \infty)$ is a normalizing constant so chosen that

$$\int_0^\infty f_\beta(t) p_t(x) dt = |x|^{-\beta} = h_\beta(x), \quad x \in \mathbb{R}^d.$$

By [6], $P_t h_\beta \le h_\beta$ (superharmonic!). For $\beta \in (0, d-\alpha)$ we also let

$$q_\beta(x) := \frac{1}{h_\beta(x)} \int_0^\infty f'_\beta(t) p_t(x) dt, \quad x \in \mathbb{R}^d.$$

By [6], $q_\beta(x) = \kappa_\beta |x|^{-\alpha}$, and $\tilde{P}_t h_\beta \le h_\beta$.

Insights for Theorem (2)

Let

$$u(x) := |x|^{-\delta/(p-1)} \wedge |x|^{-\delta}, \quad x \in \mathbb{R}^d.$$

The function "reverses" the Hardy inequality in $L^p(\mathbb{R}^d)$ with κ_δ if $\kappa_\delta > \kappa_{(d-\alpha)/p}$.

We face annoying integrability issues for u and puzzling questions about the natural domain of $\mathcal{E}_p$.

Insights for Theorem (3)

$\tilde{P}_t \sim \Delta^{\alpha/2} + \kappa|x|^{-\alpha} =: \Delta^{\alpha/2} + q$ is given by perturbation series.

For f in the domain of $\Delta^{\alpha/2}$ on $L^p(\mathbb{R}^d)$, let $u(t,x) = \tilde{P}_t f(x)$.

Then $(p > 1)$,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} |u(t)|^p dx &= \int_{\mathbb{R}^d} \frac{d}{dt}|u(t)|^p dx = \int_{\mathbb{R}^d} pu(t)^{\langle p-1\rangle}\frac{d}{dt}u(t)dx \\ &= p\int_{\mathbb{R}^d} u(t)^{\langle p-1\rangle}(\Delta^{\alpha/2}+q)u(t)dx \\ &= p\left(-\mathcal{E}_p[u(t)] + \int_{\mathbb{R}^d} q|u(t)|^p dx\right) \le 0,\end{aligned}$$

provided $\kappa \le \kappa_{(d-\alpha)/p}$.

Insight for Theorem (4)

For $\tilde{p}_t(x,y) \sim \Delta^{\alpha/2} + \kappa_\delta |x|^{-\alpha}$, where $\delta \in [0,(d-\alpha)/2]$, we have

$$\tilde{p}_t(x,y) \approx \left(1 + t^{\delta/\alpha}|x|^{-\delta}\right)\left(1 + t^{\delta/\alpha}|y|^{-\delta}\right)\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right),$$

for all $x, y \in \mathbb{R}^d$, $t > 0$. The result is given in [10].

The boundedness of $\tilde{P}_t$ on $L^p(\mathbb{R}^d)$ follows quite directly – it is characterized by $\delta \le d/p^*$, where $p^* = \max\{p, p/(p-1)\}$.

Note that $\tilde{P}_t$ is bounded on $L^2(\mathbb{R}^d)$ if $0 \le \delta \le (d-\alpha)/2$, and $\tilde{p}(x,y) = \infty$ for $\kappa \ge \kappa_{(d-\alpha)/2}$.

We have only discussed $d > \alpha$, $\kappa \ge 0$ and $p \in (1,\infty)$...

Some more insights

Note/recall that for, e.g., $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\mathcal{E}_p[\phi] = -\int_{\mathbb{R}^d} \phi(x)^{\langle p-1\rangle} \Delta^{\alpha/2}\phi(x)\,\mathrm{d}x.$$

On the other hand,

$$\mathcal{E}_p[u] \approx \mathcal{E}[u^{\langle p/2\rangle}],$$

but this may be a mouse trap, resulting in loss of accuracy/insight.

It seems that even the symmetrization,

$$\frac{1}{2}(F_p(a,b) + F_p(b,a)) = \frac{p}{2}(b-a)(b^{\langle p-1\rangle} - a^{\langle p-1\rangle}),$$

should be avoided early on.

Connections

Davies [19] and Bakry [3] give some essential calculations with forms and powers.

That $\mathcal{E}_p$ captures the evolution of the L^p norm of functions upon the action of operator semigroups is known since Varopoulos [39].

The comparison of $\mathcal{E}_p[u]$ and $\mathcal{E}[u^{\langle p/2\rangle}]$ can be traced back to Liskevich et al. [33] and [32]. See also [39], [4], Stroock [38] and Carlen, Kusuoka and Stroock [17] for formulations with nonnegative arguments or one-sided comparison.

Liskevich and Semenov [34] use the L^p setting to analyze perturbations of Markovian semigroups.

See Pinchover, Tertikas, Tintarev [35] for estimates and applications of F_p, also higher dimensions.

Connections and Bibliography

For the semigroups of local generators see Langer and Maz'ya [31] and Sobol and Vogt [36].

For nonlocal operators and bivariate forms see Farkas, Jacob and Schilling [20], Jacob [27] and Hoh and Jacob [26].

See Kinzebulatov and Semenov [28] for recent developments.

For probability connection, in particular martingale connections see KB, Dyda and Luks [7], KB and Więcek [16] and KB, Grzywny, Pietruska-Pałuba and Rutkowski [11].

The paper [11] gives related trace and extension results for the Dirichlet problem for nonlocal operators in the setting of L^p spaces.

(Still some time?) Ikeda-Watanabe and Dynkin formulas

Ikeda-Watanabe formula: for $J \subset \mathbb{R}$, $A \subset D$, $B \subset (\overline{D})^c$,

$$\mathbb{P}^x[\tau_D \in J, X_{\tau_D -} \in A, X_{\tau_D} \in B] = \int\limits_J \int\limits_B \int\limits_A p_u^D(x,y)\nu(y,z)\mathrm{d}y\mathrm{d}z\mathrm{d}u.$$

I-W gives the law of $(\tau_D, X_{\tau_D -}, X_{\tau_D})$ on $\{X_{\tau_D -} \in D\}$.

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Consider nice $U \subset\subset D$ and $\phi : \mathbb{R}^d \to \mathbb{R}$, say, C^2. Then for $x \in U$,

$$\int \phi(y)\omega_U^x(\mathrm{d}y) = \int_D \phi(z)P_D(x,z)\mathrm{d}z = \mathbb{E}^x \phi(X_{\tau_U})$$

$$\stackrel{Dynkin}{=} \phi(x) + \mathbb{E}^x \int_0^{\tau_U} L\phi(X_t)\mathrm{d}t = \phi(x) + \int_U G_U(x,y)L\phi(y)\mathrm{d}y\,.$$

Say, L is a unimodal operator with scaling and C^2 Lévy measure, or just let $L + \Delta^{\alpha/2}$.

Hardy-Stein formula (explanation)

Recal that $u : \mathbb{R}^d \to \mathbb{R}$ is L-harmonic in D if for all open $U \subset\subset D$,

$$u(x) = \mathbb{E}^x u(X_{\tau_U}) \,, \quad x \in U \,.$$

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Using ν'' and Grzywny and Kwaśnicki [23] we get

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If u is L-harmonic on D, then $u \in C^2(D)$ and $Lu = 0$ on D.

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$$Lu^2(y) = Lu^2(y) - 2u(y)Lu(y) = \int_{\mathbb{R}^d} (u(z) - u(y))^2 \nu(z,y)\,\mathrm{d}z,$$

for $y \in U$.

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for $y \in U$. Applying Dynkin to $u(x)^2$, we get Hardy-Stein:

$$\mathbb{E}^x u(X_{\tau_U})^2 = u(x)^2 + \int_U G_U(x,y) \int_{\mathbb{R}^d} (u(z) - u(y))^2 \nu(z,y) \mathrm{d}z \mathrm{d}y.$$

Some insights: Nonlinear Hardy-Stein

Recall that $F_p(a,b) = |b|^p - |a|^p - pa^{<p-1>}(b-a)$, $a, b \in \mathbb{R}$.
Since u is L-harmonic,

$$L|u|^p(y) = L|u|^p(y) - pu(y)^{\langle p-1\rangle} Lu(y)$$
$$= \lim_{\epsilon\to 0+} \int_{|z-y|>\epsilon} (|u(z)|^p - |u(y)|^p - pu(y)^{\langle p-1\rangle}(u(z)-u(y)))\nu(y,z)\,\mathrm{d}z$$
$$= \int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z).$$

To get Hardy-Stein identity we use the Dynkin formula for $|u(x)|^p$:

Lemma ([11]; for $\Delta^{\alpha/2}$ see [7])

If $u = P_D[g]$ *and* $x \in D$*, then* $\int_{D^c} |g(z)|^p \, P_D(x,z)\mathrm{d}z$ *equals*

$$|u(x)|^p + \int_D G_D(x,y) \int_{\mathbb{R}^d} F_p(u(y),u(z))\nu(y,z)\,\mathrm{d}z\mathrm{d}y.$$

Some more insights

There is a Douglas identity in L^p, proved by Hardy-Stein, mysterious cancellations and the following

Lemma

Let X be a random variable with $\mathbb{E}|X| < \infty$. Then,

$$\mathbb{E}F_p(\mathbb{E}X, X) = \mathbb{E}|X|^p - |\mathbb{E}X|^p \geq 0,$$

and

$$\mathbb{E}F_p(a, X) = F_p(a, \mathbb{E}X) + \mathbb{E}F_p(\mathbb{E}X, X), \quad a \in \mathbb{R}.$$

Note that

$$\mathcal{E}_D^{(p)}[u] \approx \mathcal{E}_D(u^{<p/2>}, u^{<p/2>}),$$

however our nonlinear Douglas identity is an exact equality [12], [8], discussed by Katarzyna Pietruska-Pałuba on Monday. See also [2], [13] for Hardy-Stein for semigroups.

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