Power of Discrete Nonuniformity – Optimizing Access to Shared Radio Channel in Ad Hoc Networks

Jacek Cichoń  Mirosław Kutyłowski  Marcin Zawada

Institute of Mathematics and Computer Science
Wrocław University of Technology
Poland

MSN’08
Contents:

1 Old protocols
   Nakano-Olariu protocol
   Cai-Lu-Wang Protocol

2 Our New Solution
   Description
   Analysis

3 Comparison of protocols

4 Conclusions and Hypothesis
Description of problem

Problem

- Consider single-hop, ad hoc network with \( n \)-stations
- There is one additional node called coordinator
- We want to choose a unique station
- Stations may transmit and listen using common radio-channel
- Stations can recognize: IDLE, SINGLE, COLISION
Common parameters

Parameters

- $\lambda$ - maximal transmission delay
- $\delta$ - the length of the shortest message

Some calculations:

- distance $d = 3$ [km], light speed $c = 3 \cdot 10^5$ [km/sec]:
  \[ \lambda \approx \frac{1}{10^5} \text{ [sec]} \]
- transmission speed 1 [Mb/sec], length 128 bits: $\delta \approx \frac{1}{10^4}$ [sec]
There are $n$ stations. Time is divided into small slots.

1. each station generates $\xi = \text{random}()$;
2. if $\xi < \frac{1}{n}$ then it transmits a message of length $\delta$;
3. If only one station transmit then coordinator sends a message OK
   else coordinator sends a message CONTINUE;
Nakano-Olariu: Analysis

Analysis

1. length of one slot: \((\delta + \lambda) + (\delta + \lambda)\)
2. probability of success in one slot \(p = \left(\frac{n}{1}\right) \frac{1}{n} (1 - \frac{1}{n})^{n-1} \sim \frac{1}{e}\)
3. Fact: expected value of a random variable with geometric distribution with parameter \(p\) is \(\frac{1}{p}\).

Theorem

Let \(NO_n\) be the length of the Nakano-Olariu leader election protocol. Then

\[
E[NO_n] \approx 2 \cdot e(\lambda + \delta) \approx 5.43656 \cdot \delta + 5.43656 \cdot \lambda.
\]
Fix probability $p$ and a time-slot $[0, T]$.

**Basic idea**

1. each station generates $\xi = \text{random}()$;
2. if $\xi < p$ then
   1. a station choose a random time $t \in [0, T]$
   2. if at time $t$ channel is idle then it starts a transmission to the end of the slot $[0, T]$
3. if in the interval $[0, T]$ there was no collision then coordinator send message OK
   else sends message CONTINUE

What is the optimal pair $(p^*, T^*)$?
### Collision

1. Let $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be moments chosen by stations.
2. There is no collision, if $X_{2:n} - X_{1:n} > \lambda$.
3. $\Pr[X_{2:n} - X_{1:n} > \lambda] = (1 - \frac{\lambda}{T})^n$. 

$\lambda$
Let $CLW_{p,T} = (\text{random variable})$ the time necessary to choose a leader in this algorithm. Then $E[CLW_{p,T}] =$

$$
\frac{T + 2(\delta + \lambda)}{Np(1 - p)^{N-1} + 1_{\lambda \leq T} \left( \sum_{k=2}^{N} \binom{N}{k} \left( (1 - \lambda / T)p \right)^k (1 - p)^{n-k} \right)}
$$

1. The behavior of this algorithm depends on a proper setting of parameters $p$ and $T$ for given $\lambda$, $\delta$ and $n$.

2. No analytical formula for the optimal choice of parameters is known.
Cai-Lu-Wang: Analysis 3

\[ \text{clw}(\delta, \lambda) = \min\{E[\text{CLW}_{p,\delta,\lambda,\tau}] : 0 \leq p \leq 1 \land T \geq 1\} . \]

1. If \( \delta < 3.1 \cdot \lambda \), then Nakano-Olariu is better than Cai-Lu-Wang
2. If \( \delta > 3.1 \cdot \lambda \) then Cai-Lu-Wang better than Nakano-Olariu
3. If \( \delta \in [\lambda, 200 \cdot \lambda] \) then

\[ \text{clw}(\delta, \lambda) \approx 3.89456 \cdot \lambda + 2.25012 \cdot \delta + 8.28525 \cdot \lambda \cdot \ln \frac{\delta}{\lambda} . \]
New solution: CKZ Protocol

Description

1. Fix $k$ and time-points $0, \lambda, 2\lambda, \ldots, (k-1)\lambda$.
2. Fix probabilities $p_0, p_1, p_2, \ldots, p_{k-1}$, such that $p_0 + \ldots + p_{k-1} \leq 1$.
3. Each station choose one of the time-point with probability $p_i$.
4. Station start transmission at chosen point if the channel is idle.
Problem: given $N$, $\lambda$, $\delta$, $k$, find optimal probabilities $p_0, p_1, p_2, \ldots, p_{k-1}$.

1. Probability of the success is

$$\sum_{i=1}^{k} \binom{N}{1} p_i (1 - (p_1 + \ldots + p_i))^{N-1}$$

2. Good approximation ($p_i = a_i/N$)

$$f_k(a_1, \ldots, a_k) = \sum_{i=1}^{k} a_i e^{-(a_1+\ldots+a_i)}$$

3. So we need to find the maximum of the function $f_k$
Recurrence

\[
\begin{cases}
M_0 &= 0 \\
M_{k+1} &= e^{-1+M_k}
\end{cases}
\]

Theorem

1. **Extremal point of** \( f_k \)

   \[
   (1 - M_{k-1}, 1 - M_{k-2}, \ldots, 1 - M_1, 1 - M_0)
   \]

2. **Maximal value of** \( f_k \) **is** \( M_k \)
Recurrence

\[
\begin{align*}
M_0 &= 0 \\
M_{k+1} &= e^{-1} + M_k
\end{align*}
\]

Properties of sequence

First five values of the sequence \((M_k)_{k \geq 0}\):

\[
0, 1/e, e^{-1} + 1/e, e^{-1} + e^{-1} + 1/e, e^{-1} + e^{-1} + e^{-1} + 1/e
\]

which are approximately equal to

\[
0, 0.367879, 0.531464, 0.625918, 0.68792
\]
Recurrence

\[
\begin{cases}
M_0 = 0 \\
M_{k+1} = e^{-1 + M_k}
\end{cases}
\]

Theorem

*The sequence \((M_k)\) is monotonically convergent to 1. Moreover*

\[
M_k = 1 - \frac{2}{k} + \frac{(2/3) \ln k}{k^2} + o \left( \frac{\ln k}{k^2} \right).
\]
Let $CKZ_k$ = our protocol with probabilities

$$(p_1, \ldots, p_k) = \left( \frac{1 - M_{k-1}}{N}, \ldots, \frac{1 - M_1}{N}, \frac{1}{N} \right).$$

Abusing notation: $CKZ_k$ = (random variable) the time necessary to choose a leader in this protocol.

**Theorem**

*For each $k \geq 1$ we have*

$$E[CKZ_k] \approx \frac{2\delta + (k + 2)\lambda}{M_k}.$$ 

*Moreover, for large $k$ we have* $E[CKZ_k] \approx 2\delta + (k + 3)\lambda$. 
All together

- Nakano-Olariu: $\frac{1}{n}$
- Cai-Lu-Wang: uniform distribution
- CKZ: $\frac{0.312}{n}$, $\frac{0.374}{n}$, $\frac{0.468}{n}$, $\frac{0.632}{n}$, $\frac{1}{n}$
Comparison with CLW protocol

Last problem: given $\lambda$, $\delta$ find optimal $k$. For $\delta \geq \lambda$ we have

$$k_{opt} \approx 2 \sqrt{\frac{\delta}{\lambda}}.$$ 

<table>
<thead>
<tr>
<th>$\delta / \lambda$</th>
<th>$CLW$</th>
<th>$CKZ_{opt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$13.5234 \cdot \lambda$</td>
<td>$9.4079 \cdot \lambda$</td>
</tr>
<tr>
<td>10</td>
<td>$44.0127 \cdot \lambda$</td>
<td>$35.301 \cdot \lambda$</td>
</tr>
<tr>
<td>50</td>
<td>$148.6400 \cdot \lambda$</td>
<td>$130.610 \cdot \lambda$</td>
</tr>
<tr>
<td>100</td>
<td>$268.0110 \cdot \lambda$</td>
<td>$242.196 \cdot \lambda$</td>
</tr>
</tbody>
</table>

Table: Expected run-times for $N = 100$
Summary

Conclusions

1. There are combinations of the Nakano-Olariu protocol and Cai-Lu-Wang protocol which improve the run-time over both of them.

2. There are precise analytical formulas for the optimal choice of parameters controlling behavior of our protocols.

3. Our protocol can be easily transformed into initialization algorithms.

4. The strategies can be adapted to the case of unknown number of stations.
The task of initializing an $n$-station with known $n$ terminates, with probability exceeding $1 - \frac{1}{n}$, in $\frac{1}{M_k} n + O(\sqrt{n \log n})$ time slots.

Therefore the task of initializing an $n$-station with known $n$ terminates, with probability exceeding $1 - \frac{1}{n}$, in

$$(1 + \frac{2}{k})n + O(\sqrt{n \log n})$$

time slots. Let us recall that for the original Nakano-Olariu protocol the bound is

$$e \cdot n + O(\sqrt{n \log n})$$.
Hypothesis

Our solution of the initialization problem is optimal in the class of algorithms which choose one station during one round in a single-hop environment.