BEYOND HYPERBOLIC DYNAMICS

C I R M, LUMINY, June 2011

Boyle–Downarowicz Lectures

5. Entropy structure in general systems

Let (X, T) be a topological dynamical system system (continuous transformation of a compact metric space).

We will define two notions of *entropy of a measure with respect to a topological resolution.*

They will be an example of "nice cooperation": one of the above notions enjoys a number of useful properties, not very hard to prove, which usually fail for the other notion. The other notion is, in turn, useful (possible to evaluate) in many applications.

Later we will show that these two notions produce two uniformly equivalent nets of functions defined on the simplex $\mathcal{M}_T(X)$.

Via uniform equivalence, consequences of the "good properties" of the first notion apply to both nets.

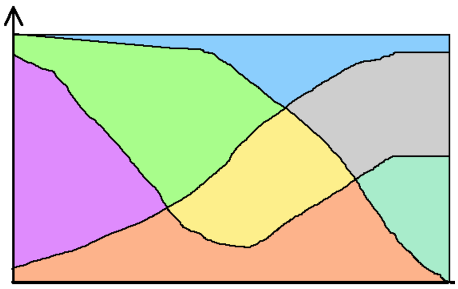
Entropy with respect to a family of continuous functions

In a zero-dimensional space, given a clopen partition $\mathcal{P}$, the characteristic functions of the "cells" $A \in \mathcal{P}$ are continuous. This is the reason why the functions $\mu \mapsto H(\mu, \mathcal{P})$ and $\mu \mapsto H(\mu, \mathcal{P}|\mathcal{Q})$ is continuous on probability masures, and $\mu \mapsto h(\mu, \mathcal{P})$ and $\mu \mapsto h(\mu, \mathcal{P}|\mathcal{Q})$ are upper semicontinuous on invariant measures.

To obtain equally good properties in general spaces, we must replace partitions by families of continuous functions.

Definition 8. Let $f : X \to [0,1]$ be continuous. We let $\mathcal{A}_f$ be the partition of $X \times [0,1]$ int the sets "above" and "below" the graph of f. For a finite family $\mathcal{F}$ of continuous functions f as above we set

$$\mathcal{A}_{\mathcal{F}} = \bigvee_{f \in \mathcal{F}} \mathcal{F}_f.$$



We then define

$$h(\mu, \mathcal{F}) = h(\mu \times \lambda, \mathcal{A}_{\mathcal{F}}),$$

where λ is the Lebesgue measure on $[0,1]$ and the action in $X \times [0,1]$ is by of $T \times \mathrm{id}$.

Finite families $\mathcal{F}$ form a directed family (by inclusion), so we have a net of functions

$$\mathcal{H}^{\text{fun}} = \{h_{\mathcal{F}}\},$$

where $h_{\mathcal{F}}(\mu) = h(\mu, \mathcal{F})$.

The following properties of this net are obvious:

• It increases,

because $\mathcal{F}' \supset \mathcal{F} \implies \mathcal{A}_{\mathcal{F}'} \succcurlyeq \mathcal{A}_{\mathcal{F}}$

• The limit is the entropy function $\mu \mapsto h(\mu)$,

because the partitions $\mathcal{A}_{\mathcal{F}}$ generate the sigma-algebra in $X \times [0,1]$ and $h(\mu \times \lambda)$ (with respect to $T \times \mathrm{id}$) is the same as $h(\mu)$ (with respect to T)

• The functions $h_{\mathcal{F}}$ and differences $h_{\mathcal{F}'} - h_{\mathcal{F}}$ (for $\mathcal{F}' \supset \mathcal{F}$) are upper semicontinuous,

because this is the usual entropy and conditinal entropy wrt. to "almost clopen" partitions – the boundaries have zero measure for every measure of the form $\mu \times \lambda$.

• Each function $h_{\mathcal{F}}$ is affine,

because this is the usual entropy function wrt. to a partition.

Conclusion: $\mathcal{H}^{\text{fun}}$ is a u.s.d.a.-net

Newhouse local entropy

This notion of "local entropy" was introduced by Sheldon Newhouse in 1989. It has two parameters: the measure and an open cover.

Definition 9. Let $\mathcal{U}$ be an open cover and let F denote a measurable set. We abbreviate $\mathcal{U}^n = \bigvee_{i=0}^{n} T^{-i}(\mathcal{U})$. We define successively:

(a) $\mathbf{H}(\delta|F,\mathcal{U}) := \log \max\{\#E : E \text{ is } (n,\delta)\text{-separated in } F \cap U, U \in \mathcal{U}\}$;

(b) $h(\delta|F,\mathcal{U}) := \limsup_n \frac{1}{n} \mathbf{H}(\delta|F,\mathcal{U}^n)$;

(c) $h(T|F,\mathcal{U}) := \lim_{\delta \to 0} h(\delta|F,\mathcal{U})$;

(d) $h(T|\mu,\mathcal{U}) := \lim_{\sigma \to 1} \inf\{h(T|F,\mathcal{U}) : \mu(F) > \sigma\}$.

We apply (d) to ergodic measures μ, then we extend the function $\mu \mapsto h(T|\mu,\mathcal{U})$ to all of $\mathcal{M}_T(X)$ by averaging over the ergodic decomposition (!)

Open covers ordered by $\succcurlyeq$ form a directed family, so we have defined a net of functions of the invariant measure, indexed by open covers. It is clear that $h(T|\mu,\mathcal{U})$ decreases with $\mathcal{U}$, so we will denote

$$\theta_{\mathcal{U}}(\mu) = h(T|\mu,\mathcal{U}) \quad \text{and} \quad \mathcal{H}^{\text{New}} = \{h - \theta_{\mathcal{U}}\}$$

although it is not at all obvious that the first net decreases to zero. Semicontinuity of $h(T|\mu,\mathcal{U})$ (or the differences) probably does not hold. In fact, for a long time it was not known whether these functions were measurable. Thus, in the averaging definition for non-ergodic measure, we have used upper integral. Hence we did not even know for sure whether the functions were affine. But... it was no problem at all! All the missing properties will be (approximatively) satisfied due to uniform equivalence with the preceding net.

Strategy

Further strategy is to

- show that the nets $\mathcal{H}^{\text{fun}}$ and $\mathcal{H}^{\text{New}}$ are preserved by principal extensions;

- relate these nets in zero-dimensional systems to more familiar functions;

- use principal zero-dimensional extensions to deduce that in general systems both nets are uniformly equivalent to each-other and that they determine symbolic extension entropies just like in the zero-dimensional case.

Properties of the two notions

Theorem 9: Let (X',T') be a topological extension of (X,T), $\mathcal{F}$, $\mathcal{U}$, $\mathcal{F}'$ and $\mathcal{U}'$ denote a finite family of functions on X, a finite open cover of X, and their respective lifts to X'. Let μ' and μ denote an invariant measure on X' and its image on X. Then

$$h(\mu',\mathcal{F}') = h(\mu,\mathcal{F})$$

and

$$h(T|\mu,\mathcal{U}) \leq h(T'|\mu',\mathcal{U}') \leq h(T|\mu,\mathcal{U}) + \mathbf{h}_{\mathrm{top}}(T'|T),$$

in particular, for principal extensions, $h(T'|\mu',\mathcal{U}') = h(T|\mu,\mathcal{U})$.

Lemma 1: Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of X. Then, for any $\mu \in \mathcal{M}_T(X)$ we have

$$h(T|\mu,\mathcal{V}) \leq h(T|\mu,\mathcal{U}) + \mathbf{h}_{\mathrm{top}}(T,\mathcal{U}|\mathcal{V}).$$

Proof: For any $\delta > 0$ and a measurable set F we have

$$\mathbf{H}(\delta|F,\mathcal{V}) \leq \mathbf{H}(\delta|F,\mathcal{U}) + \mathbf{H}(\mathcal{U}|\mathcal{V}).$$

We apply the above to $\mathcal{V}^n$ and $\mathcal{U}^n$, divide by n, pass to $\limsup$ over n, take the limit as $\delta \to 0$, and then the infimum over all sets F with large measure. $\square$

In zero-dimensional systems

Theorem 10: Let (X, T) be a zero-dimensional system. Let $\mathcal{U}$ denote a finite cover of X by disjoint clopen sets and let $\mathcal{P}$ denote $\mathcal{U}$ treated as a partition. Let $\mathcal{F}_{\mathcal{U}}$ be the family of characteristic functions of the cells of $\mathcal{U}$. Then

$$h(\mu, \mathcal{F}_{\mathcal{U}}) = h(\mu, \mathcal{P}) \quad \text{and} \quad h(T|\mu, \mathcal{U}) = h(\mu|\mathcal{P}) = h(\mu) - h(\mu, \mathcal{P}).$$

Proof: The first equality is obvious. The second is much harder. It uses the Shannon–McMillan–Breiman Theorem.

Theorem 11: Let (X,T) be a zero-dimensional system. The nets $\mathcal{H}^{\text{fun}}$ and $\mathcal{H}^{\text{New}}$ are uniformly equivalent to the "standard" entropy structure $\mathcal{H} = \{h_k\}$, where $h_k(\mu) = h(\mu, \mathcal{P}_k)$ for a refining sequence of clopen partitions $\mathcal{P}_k$.

(The "standard" entropy structure in dimension zero was introduced by Mike in his talks.)

Proof: The partitions $\mathcal{P}_k$, treated as as open covers $\mathcal{U}_k$, form a sequence which is a subnet of the net of all covers $\mathcal{U}$. Their corresponding families $\mathcal{F}_{\mathcal{U}_k}$ of characteristic functions is a sub-net of the net $\mathcal{F}$. By the preceding theorem, we obtain that $\mathcal{H}$ is a subnet of $\mathcal{H}^{\text{New}}$, hence these two are uniformly equivalent.

Also, we obtain that $\mathcal{H}$ is a sub-net of $\mathcal{H}^{\text{fun}}$. But since the latter net has upper semicontinuous differences and both nets converge to the entropy function, yesterday's Theorem 6 implies they are uniformly equivalent. $\square$

From zero-dimensional to general

Theorem 12: Let (X', T') be a principal extension of (X, T). Then the nets $\mathcal{H}^{\text{fun}}$ and $\mathcal{H}^{\text{New}}$ defined for the system (X, T) and lifted to $\mathcal{M}_{T'}(X')$ are uniformly equivalent to the nets $\mathcal{H}'^{\text{fun}}$ and $\mathcal{H}'^{\text{New}}$ defined for the system (X', T'), respectively.

Proof: The proof is easy for $\mathcal{H}^{\text{fun}}$. By Theorem 9, this net, lifted, becomes a sub-net of $\mathcal{H}'^{\text{fun}}$. Since the extension is principal, so obtained sub-net has the same limit function. Yesterday's Theorem 6 completes the proof.

For $\mathcal{H}^{\text{New}}$, Theorem 9 also implies that this net, lifted, becomes a sub-net of $\mathcal{H}'^{\text{New}}$. So it is uniformly dominated. For the converse domination, let $\mathcal{V}'$ be a cover of X'. Since the extension is principal, $\mathbf{h}_{\text{top}}(T'|T) = 0$, which means that for every cover of X' (in particular for $\mathcal{V}'$), there exists an open cover $\mathcal{U}$ of X, such that $\mathbf{h}_{\text{top}}(T', \mathcal{V}'|\mathcal{U}') < \epsilon$, where $\mathcal{U}'$ is the lift of $\mathcal{U}$. Then, by Lemma 1,

$$h(T'|\mu', \mathcal{U}') \leq h(T'|\mu', \mathcal{V}') + \mathbf{h}_{\text{top}}(T', \mathcal{V}'|\mathcal{U}'),$$

i.e.,

$$h(\mu') - h(T|\mu', \mathcal{U}') \geq h(\mu') - h(T'|\mu', \mathcal{V}') - \epsilon.$$

□

Entropy structure in general systems

We are in a position to prove a theorem allowing to introduce the key notion of the theory.

Theorem 13: Let (X,T) be an arbitrary system. The nets $\mathcal{H}^{\text{fun}}$ and $\mathcal{H}^{\text{New}}$ are uniformly equivalent to each-other.

Proof: This is now an immediate consequence of the preceding two theorems. $\Box$

Definition 10: The **entropy structure** of (X,T) is defined as the uniform equivalence class containing $\mathcal{H}^{\text{fun}}$ and $\mathcal{H}^{\text{New}}$.
The term *entropy structure* will also denote each element of this equivalence class.

Remark: Many other familiar entropy notions with a topological parameter belong here. For example, the Katok entropy of a measure computed by counting (n,ϵ)-balls needed to cover a set of certain positive measure, the Brin-Katok entropy imitating the S-M-B Theorem for the (n,ϵ)-balls, a version of the Ornstein–Weiss entropy estimate based on the first return time to the (n,ϵ)-ball, and more recently, Romagnoli's entropy of a measure given an open cover using partitions inscribed in the given cover.

Symbolic extension entropy theorem - general

Theorem: Let (X, T) be a dynamical system. Let E be a function defined on $\mathcal{M}_T(X)$.

TFAE:

(1) E is a superenvelope of the entropy structure

(2) $E = h_{\text{ext}}^{\varphi}$ for a symbolic extension $\varphi : (Y, S) \to (X, T)$.

In particular $h_{\text{sex}} = E_{\min}$ and $\mathbf{h}_{\text{sex}} = \min_{\mu \in \mathcal{M}_T(X)} E(\mu)$.

There exists a symbolic extension with $h_{\text{ext}}^{\varphi} = E_{\min}$ if and only if $E_{\min}$ is affine (for example, when $E_{\min} = h$, i.e., in the asymptotically h-expansive case).

What is entropy structure good for

Entropy structure is a *master entropy invariant*, allowing to derive most of other entropy invariants:

• The entropy function h, as the limit of $\mathcal{H}$.

• Topological entropy, as $\sup_\mu h(\mu) = \lim_\kappa \uparrow \sup_\mu h_\kappa(\mu)$.

(Implicitly, entropy structure has been used in this role for years.)

• Symbolic extension entropies:
∘ h^φ_{ext} in symbolic extensions (Mike) as affine superenvelopes of $\mathcal{H}$,
∘ h_{sex} as $E_{\min}$ (because for u.s.d.a.-nets $E_{\min} = \inf E_A$),
∘ the *residual entropy function* $h_{\text{res}} = h - h_{\text{sex}}$ as $u_{\min}$,
∘ $\mathbf{h}_{\text{sex}}(T)$ as $\sup_\mu E_{\min}(\mu)$ (because $\min\{\sup_\mu E_A(\mu)\} = \sup_\mu E_{\min}(\mu)$),
∘ $\mathbf{h}_{\text{res}}(T) = \mathbf{h}_{\text{sex}}(T) - \mathbf{h}_{\text{top}}(T)$ as $\sup_\mu h(\mu) - \sup_\mu E_{\min}(\mu)$
(there is no such thing as *residual variational principle* – this is the reason why we do not use residual entropy so much).

• Entropy structure allows to decide when is h_{sex} realized as h^φ_{ext} for a symbolic extension. It is so if and only if h_{sex} is affine.

• Also the *topological tail entropy* $\mathbf{h}^*(T)$ (known as Misiurewicz's topological conditional entropy) equals $\sup_\mu u_1(\mu)$, which equals $\sup_\mu D_\mu$, which equals the global defect of the uniform convergence of $\mathcal{H}$ (recall the beginning of lecture 3).

This last equality is a fairly new discovery (D. & Burguet). It provides a number of charcterizations of *asymptotically h-expansive* systems (by definition, with $\mathbf{h}^*(T) = 0$):

• TFAE: the system is asymptotically h-expansive, $u_1 = 0$, $\alpha_0 = 0$, $\mathcal{H} \rightrightarrows h$, $h_{\text{sex}} = h$, **there exists a principal symbolic extension**.

Finally, entropy structure introduces some new entropy invariants, unknown before:

• The *transfinite sequence* (u_α) of the entropy structure,

• The *order of accumulation* α_0 of the entropy structure, and more (Burguet, McGoff).